

The cobordism category and Waldhausen's K -theory.

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0 Introduction

This paper examines the category $\mathcal{C}_{d,n}^k$ whose morphisms are d -dimensional smooth manifolds that are properly embedded in $I^k \times \mathbb{R}^{d+n-k}$, where I^k is a k -dimensional cube. There are k directions to compose k -dimensional cubes, so $\mathcal{C}_{d,n}^k$ is a (strict) k -tuple category. The geometric realization of the k -dimensional multi-nerve $N_\bullet \mathcal{C}_{d,n}^k$ is the classifying space $BC_{d,n}^k$. Its homotopy type is determined by theorem 2.8 below to be

$$BC_{d,n}^k \simeq \Omega^{d+n-k} \text{Th}(U_{d,n}^\perp) \quad (1)$$

where $U_{d,n}^\perp$ is the n -dimensional canonical bundle over the Grassmannian $G(d, n)$ of d -planes in \mathbb{R}^{n+d} and where Th denotes the Thom space, that is the one point compactification of $U_{d,n}^\perp$.

For $k = 1$ and $n = \infty$, the structure of $BC_{d,n}^k$ was determined in [6], using the sheaf techniques of [15]. We note from (1) that

$$\Omega BC_{d,n}^k \simeq BC_{d,n}^{k-1}, \quad k \leq d + n.$$

For $n \rightarrow \infty$ we get a geometric interpretation of the Ω -spectrum $MTO(d)$ of [6], namely

$$MTO(d) \cong \{BC_{d,\infty}^k\}_{k=1}^\infty. \quad (2)$$

At the end of the paper in §5 we use (2) to construct an infinite loop map

$$\Omega BC_{d,n}^1 \rightarrow A(BO(d)) \quad (3)$$

We believe that the map factors through $\Omega^\infty \Sigma^\infty(BO(d)_+)$ and that the composite

$$B\text{Diff}(M^d) \rightarrow \Omega BC_{d,n}^1 \rightarrow A(BO(d))$$

is homotopic to the map considered in [3].

Our method of proof of (1) is rather different from [6]. We begin with the abstract transversality theorem of §1: Given a metric space X and a closed subspace $Z \subset X \times \mathbb{R}^k$ with the property that $Z \cap (\{x\} \times \mathbb{R}^k)$ has measure 0, we introduce a simplicial space $|K_\bullet(X, Z)|$ with a map to X and show in theorem 1.8 that

$$|K_\bullet(X, Z)| \xrightarrow{\sim} \quad (4)$$

is a homotopy equivalence.

Let $\Psi_d(\mathbb{R}^{d+n})$ be the space of properly embedded d -dimensional smooth submanifolds of \mathbb{R}^{d+n} equipped with a topology where, roughly speaking, two manifolds are close if one is contained in a tubular neighbourhood of the other. For the space X in (4) we take the subspace $D_{d,n}^k \subset \Psi_d^{d+n}$ where $M \in D_{d,n}^k$ if the projection on the first k coordinates is a proper map from M to \mathbb{R}^k . The space $Z = Z_{d,n}^k \subset D_{d,n}^k \times \mathbb{R}^k$ consists of pairs (M, \underline{a}) where M fails to be transversal to the “corners” determined by $\underline{a} \in \mathbb{R}^k$.

It turns out that $K_\bullet(D_{d,n}^k, Z_{d,n}^k)$ is homotopy equivalent to the multi-nerve $N_\bullet(\mathcal{C}_{d,n}^k)$, so by (4) $B(\mathcal{C}_{d,n}^k) \simeq D_{d,n}^k$. An application of Gromov’s general h-principle yields that $D_{d,n}^k$ is homotopy equivalent to the right hand side of (1).

The proof of (4) needs that $X = D_{d,n}^k$ or $\Psi_d(\mathbb{R}^{d+n})$ be metrizable. This is proved in the technical §4. It is finally in order to remark that §3 contains results about simplicial spaces, needed in the proof of (4) which may be of general interest.

We would like to acknowledge the inspiration from S. Galatius’ manuscript [5].

1 Abstract transversality

A critical pair is a space X and a closed subset $Z \subset X \times \mathbb{R}^k$. We call Z the critical datum, and use the notation $Z(x) = \{v \in \mathbb{R}^k \mid (x, v) \in Z\}$.

Let \mathbb{R}^k be partially ordered by $\underline{a}_1 \leq \underline{a}_2$ if the inequality is valid for each coordinate, that is if $a_1^i \leq a_2^i$ for $1 \leq i \leq k$. For a totally ordered pair of vectors $\underline{a}_1 \leq \underline{a}_2 \in \mathbb{R}^k$ we consider the set of 2^k vectors obtained by mixing the coordinates, i. e. the set of 2^k vectors in the cube with southwest vertex \underline{a}_1 and northeast vertex \underline{a}_2 :

$$V(\underline{a}_1, \underline{a}_2) = \{(a_{f(1)}^1, a_{f(2)}^2, \dots, a_{f(k)}^k) \mid f: \{1, \dots, k\} \rightarrow \{1, 2\}\}.$$

Definition 1.1. A pair $\underline{a}_1 \leq \underline{a}_2$ is compatible with $Z(x) \subset \mathbb{R}^k$ if none of the vectors in $V(\underline{a}_1, \underline{a}_2)$ is contained in $Z(x)$.

In particular, if $\underline{a}_1 \leq \underline{a}_2$ are compatible, then $\underline{a}_i \notin Z(x)$.

Definition 1.2. The simplicial space of cut sets $K_\bullet(X, Z)$ is given by

$$K_q(X, Z) = \{x \in X, \underline{a}_0 \leq \underline{a}_1 \leq \dots \leq \underline{a}_q; \text{ all pairs } \underline{a}_i \leq \underline{a}_j \text{ compatible}\},$$

topologized as subspace of $X \times \mathbb{R}^{k(q+1)}$.

The simplicial space of discrete cut sets denoted by $K_q^\delta(X, Z)$ is the same underlying set but topologized as a subset of $X \times (\mathbb{R}^{k(q+1)\delta})$. That is, we do not change the topology in the X direction, but discretize the topology in the \mathbb{R}^k direction.

Let $N_\bullet(\mathbb{R}^k)$ be the nerve of the partially ordered set \mathbb{R}^k . Then $K_q(X, Z)$ is a subspace of the product $X \times N_q(\mathbb{R}^k)$, and we define the simplicial structure maps of $K_\bullet(X, Z)$ so that this is an inclusion of simplicial spaces.

The simplicial space $K_\bullet(X, Z)$ is the diagonal of a k -dimensional simplicial space, since the k -dimensional boxes of $K_1(X, Z)$ can be composed in k different directions. For $k = 2$, $\omega = (x, \underline{a}_0 \leq \underline{a}_1) \in K_1(X, Z)$ can be pictured as the square in \mathbb{R}^2 whose vertices $V(\underline{a}_0, \underline{a}_1)$ all lie outside $Z(x)$. These squares can be composed horizontally and vertically, and we let $K_{p,q}(X, Z)$ denote the grid of $p \times q$ squares all of whose vertices are outside $Z(x)$. Removing vertical and horizontal edges defines the bi-simplicial structure on $K_{\bullet,\bullet}(X, Z)$. Its diagonal simplicial space is $K_\bullet(X, Z)$. The situation is similar for $k > 2$.

Remark 1.3. One can interpret $K_\bullet(X; Z)$ as the nerve of a strict k -tuple category in the sense of [12], defined inductively as follows: A strict 0-category is a set, and a strict k -tuple category is a category object in the category of strict $k - 1$ -tuple categories. A strict 2-tuple category is a pair of small categories $\mathcal{C}_0, \mathcal{C}_1$ together with functors

$$\mathcal{C}_1 \begin{array}{c} \xrightarrow{d_1} \\ \xleftarrow{d_0} \end{array} \mathcal{C}_0 \quad \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \xrightarrow{\circ} \mathcal{C}_1,$$

In particular, the objects of \mathcal{C}_1 are the morphism of a category \mathcal{C}' , whose objects are the objects of \mathcal{C}_0 .

An element $\omega \in N_1\mathcal{C}_1 = \text{mor } \mathcal{C}$ gives rise to a square

$$\begin{array}{ccc} \underline{a}_{1,1} & \xrightarrow{g_{1,1}} & \underline{a}_{0,1} \\ \downarrow f_{1,1} & & \downarrow f_{0,1} \\ \underline{a}_{1,0} & \xrightarrow{g_{1,0}} & \underline{a}_{0,0} \end{array},$$

where $g_{1,1} = d_0\omega$, $g_{1,0} = d_1\omega$ and $\underline{a}_{i,j}$ is the source and target of the morphisms $g_{1,1}$ and $g_{1,0}$ in \mathcal{C}_0 . The vertical arrows are the objects in \mathcal{C}_1 which are the source and target of ω . They are considered as morphisms in \mathcal{C}' .

These diagrams can be composed horizontally and vertically to defines bi-simplicial set (or space, in case one deals with topological categories) $N_{\bullet,\bullet}(\mathcal{C}_1, \mathcal{C}_0)$.

In the case of a critical pair $Z \subset X \times \mathbb{R}^2$, the associated 2-tuple category is defined as follows:

$$\begin{aligned} K_0(X; Z) &= N_0\mathcal{C}_0 = X \times \mathbb{R}^2 \setminus Z, \\ N_1\mathcal{C}_0 &= \{(x, \underline{a}_1, \underline{a}_2) \in X \times \mathbb{R}^2 \times \mathbb{R}^2 \mid \underline{a}_1 \leq \underline{a}_2, \underline{a}_1^2 = \underline{a}_2^2, V(\underline{a}_1, \underline{a}_2) \cap Z(x) = \emptyset\}, \\ N_0\mathcal{C}_1 &= \{(x, \underline{a}_1, \underline{a}_2) \in X \times \mathbb{R}^2 \times \mathbb{R}^2 \mid \underline{a}_1 \leq \underline{a}_2, \underline{a}_1^1 = \underline{a}_2^1, V(\underline{a}_1, \underline{a}_2) \cap Z(x) = \emptyset\}, \\ K_1(X; Z) &= N_1\mathcal{C}_1 = \{(x, \underline{a}_1, \underline{a}_2) \in X \times \mathbb{R}^2 \times \mathbb{R}^2 \mid \underline{a}_1 \leq \underline{a}_2, V(\underline{a}_1, \underline{a}_2) \cap Z(x) = \emptyset\}. \end{aligned}$$

Then $K_{\bullet,\bullet}(X; Z) = N_{\bullet,\bullet}(\mathcal{C}_1, \mathcal{C}_0)$. The situation for $k \geq 2$ is similar, since the nerve of a simplicial object in the category of $(k - 1)$ -dimensional simplicial sets is a k -dimensional simplicial set.

Our first result is that the topology on the Euclidean factor does not matter much.

Theorem 1.4. *Assume that X is a metrizable space. The map $\lambda_Z : K_\bullet^\delta(X; Z) \rightarrow K_\bullet(X; Z)$ is a weak homotopy equivalence.*

Proof. We will consider certain subspaces of $K_\bullet(X; Z)$. For a subset $W \subset \mathbb{R}^k$, let $K_\bullet(X; Z)(W)$ consist of all simplices of $K_\bullet(X; Z)$ whose vertices are contained in W . Given a simplex $(x, \underline{a}_0, \dots, \underline{a}_r) \in K_r(X; Z)$ with $\underline{a}_0 < \dots < \underline{a}_r$, we can find an open neighbourhood $U \subset X$ of x and disjoint axis-parallel cubes W_i containing \underline{a}_i with the following properties.

- (i) The set $\{W_i\}$ is totally ordered in the sense that for $\underline{w}_i \in W_i$, $\underline{w}_0 < \dots < \underline{w}_r$.
- (ii) For $y \in U$ and $\underline{w}_0 < \dots < \underline{w}_r$ as in (i), $(y, \underline{a}_0, \dots, \underline{a}_r)$ determines a k -simplex of $K_\bullet(X; Z)$, i.e. $K_\bullet(U; Z \cap (U \times \mathbb{R}^k))(W) = K_\bullet(U; \emptyset)(W)$, where W is the disjoint union of the W_i .

This follows because the condition that $(y, \underline{a}_0, \dots, \underline{a}_r)$ belongs to $K_\bullet(U; \emptyset)(W)$ is an open condition.

We consider $W = \coprod W_i \subset \mathbb{R}^k$ and $U \subset X$ of the above type (satisfying (i) and (ii) above, and note that if (U, W) and (U, W') are two such pairs, then so is $(U, W \cap W')$.

Claim 1. If (W, U) is a pair satisfying the two conditions, the maps

$$\lambda_Z : K_\bullet^\delta(U; Z \cap (U \times \mathbb{R}^k))(W) \rightarrow K_\bullet(U; Z \cap (U \times \mathbb{R}^k))(W) \rightarrow U$$

are both homotopy equivalences. Because of the second condition, $K_\bullet(U; Z \cap (U \times \mathbb{R}^k))(W) \cong U \times K_\bullet(p, \emptyset)(W)$, where p is a one point space, so it is enough to show that both $K_\bullet(p, \emptyset)(W)$ and $K_\bullet^\delta(p, \emptyset)(W)$ are contractible.

The usual cofinality argument easily shows that $K_\bullet^\delta(p, \emptyset)(W)$ is contractible. Indeed, it's enough to show that if A is a compact space, then any map $f : A \rightarrow |K_\bullet^\delta(p, \emptyset)(W)|$ is homotopic to a constant map. But the image of a compact set will be contained in a finite simplicial subset $X_\bullet \subset K_\bullet^\delta(p, \emptyset)(W)$. The vertices of X_\bullet are given by finitely many points of W . To a finite set of points in W there is a point $\underline{b} \in W$ strictly greater than each one of them. The image of f is then contained in the cone of X_\bullet with vertex at \underline{b} , so that f is homotopic to a constant map.

Now the claim is reduced to proving that $K_\bullet(p, \emptyset)(W) = N_\bullet(W, \leq)$ is also contractible. This simplicial space is the nerve of the topological partially ordered set W . Suppose that $W = \cup_{1 \leq i \leq c} W_i$ where W_i are the components of W , in the given total order. Let $\underline{a} < \underline{b} \in \mathbb{R}^k$ be given such that the largest component of W is

$$W_c = \{\underline{x} \in \mathbb{R}^k \mid \underline{a} < \underline{x} < \underline{b}\}.$$

and let

$$W'_c = \{\underline{x} \in \mathbb{R}^k \mid \underline{a} < \underline{x} < (\underline{a} + \underline{b})/2\} \subset W_c.$$

We put $W' = (\cup_{1 \leq i \leq c-1} W_i) \cup W'_c$. Let $i : W' \rightarrow W$ be the inclusion, and let $h : W \rightarrow W'$ the map which is the identity on W_i for $i \leq c-1$, and $h(\underline{x}) = (\underline{a} + \underline{x})/2$ for $\underline{x} \in W_c$. Since

$i \circ h(\underline{x}) \leq \underline{x}$, there is a natural transformation from the functor $i \circ h$ of the category (W, \leq) to the identity, so that the identity on $K_\bullet(p, \emptyset)(W)$ factors up to homotopy over the inclusion $i : K_\bullet(p, \emptyset)(W') \rightarrow K_\bullet(p, \emptyset)(W)$. Let $w \in W$ be larger than every element in W' . The nerve of $\{W' \cup \{w\}\}$ is contractible, since it has a largest element. It follows that the composite $i : W' \subset \{W' \cup \{w\}\} \subset W$ induces a map on nerves which is homotopic to a constant map. This proves that $K_\bullet(p, \emptyset)(W)$ is contractible, and finishes the proof of claim 1.

Note that it follows trivially from this computation the natural map $K_\bullet^\delta(p, \emptyset)(W) \rightarrow K_\bullet(p, \emptyset)(W)$ is a homotopy equivalence.

Claim 2. For any finite set of such $\{W^j\}_{1 \leq j \leq n}$ and $U^j \subset X$, $j = 1, \dots, n$ which satisfy (i) and (ii), the natural map

$$\bigcup_j |K_\bullet^\delta(U^j; Z \cap (U^j \times \mathbb{R}^k))(W^j)| \rightarrow \bigcup_j |K_\bullet(U^j; Z \cap (U^j \times \mathbb{R}^k))(W^j)|$$

is a homotopy equivalence. The proof is by induction on n . The induction start is claim 1 above. Let us put $A_\bullet^j = K_\bullet(U^j; Z \cap (U^j \times \mathbb{R}^k))(W^j)$ and $A_\bullet^{j,\delta} = K_\bullet^\delta(U^j; Z \cap (U^j \times \mathbb{R}^k))(W^j)$. Consider the diagram

$$\begin{array}{ccccc} \bigcup_{1 \leq j \leq n-1} |A_\bullet^{j,\delta}| & \longleftarrow & (\bigcup_{1 \leq j \leq n-1} |A_\bullet^{j,\delta}|) \cap |A_\bullet^{n,\delta}| & \longrightarrow & |A_\bullet^{n,\delta}| \\ \downarrow & & \downarrow & & \downarrow \\ \bigcup_{1 \leq j \leq n-1} |A_\bullet^j| & \longleftarrow & (\bigcup_{1 \leq j \leq n-1} |A_\bullet^j|) \cap |A_\bullet^n| & \longrightarrow & |A_\bullet^n| \end{array}$$

By induction the left and the middle vertical maps are homotopy equivalences, and claim 1 says the right vertical map is a homotopy equivalence, so the diagram induces a homotopy equivalence from the homotopy pushout of the upper row to the homotopy pushout of the lower row. We need to show that the homotopy pushout of the rows are homotopy equivalent to the degreewise pushouts.

Degreewise, the pushout

$$\begin{array}{ccc} (\bigcup_{1 \leq j \leq n-1} A_p^j) \cap A_p^n & \longrightarrow & \bigcup_{1 \leq j \leq n-1} A_p^n \\ \downarrow & & \downarrow \\ A_p^n & \longrightarrow & \bigcup_{1 \leq j \leq n} A_p^n \end{array}$$

is a homotopy pushout, because it is the union of two open sets in a normal space (this uses the condition that X is metrizable). But the realization of a degreewise homotopy pushout diagram of simplicial spaces is a homotopy pushout, because the realization of a degreewise mapping cylinder is homeomorphic to the mapping cylinder of the realization. This concludes the induction step, and finishes the proof of claim 2.

Claim 3. Let $p \in |K_\bullet(X; Z)|$. There is a set $W^p \subset \mathbb{R}^k$ satisfying conditions (i) and (ii) above, and an open set $U^p \subset X$, such that p is in the image of the natural map

$$|K_\bullet(U^p, Z \cap (U^p \times \mathbb{R}^k))(W^p)| \rightarrow |K_\bullet(X; Z)|.$$

The point p is in the image of a characteristic map of a simplex determined by $x \in X$ and a totally ordered set of vectors $\underline{a}_0 < \underline{a}_1 < \dots < \underline{a}_r$. The point x and the vectors \underline{a}_i satisfy a number of conditions, determined by the closed set $Z \subset X \times \mathbb{R}^k$. These conditions are open conditions. This means that there is an open set $U^p \subset X$ and a set of open cubes $C_i(a) \ni \underline{a}_i$, such that $C_0 < C_1 < \dots < C_k$, and so that if we put $W^p = \cup_i C_i$ $u \in U$, then the simplex in $K_r(X; Z)$ given by $(u, \underline{b}_0 \leq \underline{b}_1 \leq \dots \leq \underline{b}_r)$ is contained in $K_r(U; Z \cap (U \times \mathbb{R}^k))$ if all $\underline{b}_j \in W^p$ and $u \in U^p$. This proves claim 3. In conclusion, we have a covering of $|K(X; Z)|$ by realizations of simplicial subsets $K_\bullet(U, Z \cap (U \times \mathbb{R}^k))$. According to claim 2, these sets have the property that the map from the corresponding discrete version is a homotopy equivalence. If these subsets were open, we would be done. In general they are not open, but they are degreewise open in the simplicial space $K_\bullet(X; Z)$. Theorem 3.5 below completes the proof. \square

We can now deal with the simplest special case.

Lemma 1.5. *Let p be a one point space, $Z \subset \mathbb{R}^k$ a closed set of measure zero. Then $|K(p; Z)|$ is weakly contractible.*

Proof. By theorem 1.4, it is sufficient to prove that $|K_\bullet^\delta(p, Z)|$ is weakly contractible. It's enough to show that the inclusion of any finite subcomplex of $|K_\bullet^\delta(p, Z)|$ is homotopic to a constant map. Any finite subcomplex will involve finitely many simplices, which are defined using a finite set of points $\underline{a} \in \mathbb{R}^k \setminus Z$.

Assume that $A = \{\underline{a}_i\}_{1 \leq i \leq n}$ is a finite subset of $\mathbb{R}^k \setminus Z$, and $i_A: G(A) \subset K_\bullet^\delta(p, Z)$ the simplicial subset consisting of all simplices of $K_\bullet^\delta(p, Z)$ with vertices in A . It suffices to show that the map i_A is homotopic to a constant map. We show inductively that for any $0 \leq m \leq n$ we can find a \underline{b} such that $\underline{a}_i \leq \underline{b}$ for $1 \leq i \leq n$, and such that \underline{a}_i and \underline{b} are compatible for $i \leq m$. It follows that $G(A \cup \{\underline{b}\})$ is contractible, since it is a cone on the vertex \underline{b} . For $m = 0$ pick any point $\underline{b} \in \mathbb{R}^k$ larger than all the \underline{a}_i .

To do the induction step, assume that we can find a \underline{b}' such that $\underline{a}_i \leq \underline{b}'$ for all i , and such that \underline{a}_i and \underline{b}' are compatible for $1 \leq i \leq m - 1$. Since Z is closed, there is an open cube centered on \underline{b}' , such that any point in this cube is also greater than all \underline{a}_i and compatible with \underline{a}_i , $1 \leq i \leq m - 1$. Since Z has measure 0, there is at least one point \underline{b} in this cube that satisfies that \underline{a}_m and \underline{b} are compatible. \square

Lemma 1.6. *Let X be a metrizable space. The simplicial space $K_\bullet = K_\bullet(X, Z)$ is a good simplicial space in the sense of [20], so that the fat realization $\|K_\bullet\|$ is homotopy equivalent to the standard realization $|K_\bullet|$.*

Proof. It suffices to show that each degeneracy map $K_p \rightarrow K_{p+1}$ is a cofibration. Let $(x, \underline{v}_0, \dots, \underline{v}_p)$ where $x \in X$ and $\underline{v}_i \in \mathbb{R}^k$. The degeneracy map s_i iterates the vector \underline{v}_i . We can identify K_p with the subset $A \subset K_{p+1}$, defined by the equation that $\underline{v}_i = \underline{v}_{i+1}$. In particular, the image of the degeneracy map is closed. Since X is metrizable, K_{p+1} is normal, and it suffices to prove that $A \subset K_{p+1}$ is a deformation retract of a neighborhood, cf [19], Satz 1.

Let $h(x, \underline{v})$ be the distance from \underline{v} to $Z(x)$. We claim that this function is upper semi-continuous, in the sense that for any $c \in \mathbb{R}$ the set $\{(x, \underline{v}) \in X \times \mathbb{R}^k \mid h(x, \underline{v}) > c\}$ is open in $X \times \mathbb{R}^k$. To see this, consider a point (x_0, \underline{v}_0) such that $h(x_0, \underline{v}_0) > c$. Choose a number q , such that $h(x_0, \underline{v}_0) > q > c$. We can cover the closed disc $\{x_0\} \times D(\underline{v}_0, q)$ with open sets in $X \times \mathbb{R}^k$, disjoint from Z . By compactness of the closed disc, there is an open neighbourhood U of x_0 in X so that $U \times D(\underline{v}_0, q)$ is disjoint from Z . By the triangle inequality, this implies that if $u \in U$ and $\underline{v} \in D(\underline{v}_0, q - c)$, then $h(u, \underline{v}) > c$. So $U \times D(\underline{v}_0, q - c)$ is neighbourhood of (x_0, \underline{v}_0) in $h^{-1}(c, \infty)$.

It follows that

$$V = \{(x, \underline{v}_1, \dots, \underline{v}_p) \in \mathbb{R}^k \setminus Z(x) \mid d(\underline{v}_i, \underline{v}_{i+1}) < d(\underline{v}_i, Z(x))/2\}$$

is open, and we can define a deformation retraction from V to A by

$$H_t(x, \underline{v}_1, \dots, \underline{v}_i, \underline{v}_{i+1}, \dots, \underline{v}_p) = (x, \underline{v}_1, \dots, \underline{v}_i, \underline{v}_{i+1} + t(\underline{v}_i - \underline{v}_{i+1}), \dots, \underline{v}_p).$$

□

The fibres of the projection $p_Z : |K_\bullet(X, Z)| \rightarrow X$ are $|K_\bullet(\{x\}, Z(x))|$ which are contractible by lemma 1.5. We need a criterion that guarantees that p_Z is a weak homotopy equivalence. The following theorem is proved in section 3 below.

Theorem 1.7. *Let X be a space, and K_\bullet a simplicial subspace of $X \times N_\bullet$ such that K_q is open in $X \times N_q$ for all q . Let $\pi : |K_\bullet| \rightarrow X$ be the projection. Suppose that N_\bullet is contractible. Then π is a weak homotopy equivalence.*

We are now ready to state the main result of this section.

Theorem 1.8. *Let (X, Z) be a critical pair, X a metrizable space, and assume that $Z(x) \subset \mathbb{R}^k$ has measure 0 for each $x \in X$. Then the projection $p_Z : |K_\bullet(X, Z)| \rightarrow X$ is a weak homotopy equivalence.*

Proof. Since $Z \subset X \times \mathbb{R}^k$ is closed, $K_q(X, Z)$ is an open subset of $K_q(X, \emptyset) = X \times N_q(\mathbb{R}^k)$. The fibre $p_Z^{-1}(x)$ can be identified with $|K_\bullet(x, Z(x))|$. By lemma 1.5 both $p_Z^{-1}(X)$ and $N_\bullet(\mathbb{R}^k) = K_\bullet(\{x\}, \emptyset)$ are weakly contractible, so theorem 1.7 applies. □

2 Categories of embedded manifolds

In this section we show how the abstract theory of section 1 applies to the theory of embedded manifolds, and we define the k -tuple category of manifolds in the Cartesian product of a euclidean space and a k -dimensional cube. We show that it deloops the category of embedded manifolds considered in [5] and [6].

2.1 The space of embedded manifolds

Following [5], we consider the space of properly embedded smooth d -manifolds without boundary in Euclidean $(d+n)$ -space,

$$\Psi_d(\mathbb{R}^{d+n}) = \{W^d \subset \mathbb{R}^{d+n} \mid \partial W = \emptyset, W \text{ a closed subset}\}.$$

We topologize $\Psi_d(\mathbb{R}^{d+n})$ so that a sequence of manifolds that leaves each compact set converges to the base point $\emptyset \in \Psi_d(\mathbb{R}^{d+n})$, and so that manifolds close to W are sections in a thin normal tube on a compact set. More precisely, let $NW \subset \mathbb{R}^{d+n}$ be a normal tube and let $K \in \mathbb{R}^{d+m}$ be a compact subset. Let $r : NW \rightarrow W$ be the projection and let C^∞ be the set of smooth sections of r . We equip it with the C^∞ -Whitney topology. For technical reasons, we first chose a metric μ on the compact Grassmannian manifold $G(d, n)$ of d -planes in $(d+n)$ -space. For $s \in C^\infty(W, NW)$ and a compact set $K \subset \mathbb{R}^{d+n}$ we define

$$\|s\|_K = \sup_{x \in W \cap K} (|s(x)| + \mu(T_x W, ds(T_x W))) ,$$

where $|\cdot|$ is the norm in \mathbb{R}^{d+n} and d denotes the differential. The open neighbourhoods of $W \in \Psi_d(\mathbb{R}^{d+n})$ are specified by a pair (K, ϵ) with K as above and $\epsilon > 0$. Define

$$\Gamma_{K, \epsilon} = \{s \in C^\infty(W, NW) \mid \|s\|_K < \epsilon\}.$$

and define the corresponding neighbourhood of W in $\Psi_d(\mathbb{R}^{d+n})$ to be

$$\mathcal{N}_{K, \epsilon}(W) = \{V^d \mid V^d \cap K = s(W) \cap K \text{ for } s \in \Gamma_{K, \epsilon}\}.$$

The neighbourhoods of the empty manifold are

$$\mathcal{N}_K(\emptyset) = \{V \in \Psi_d(\mathbb{R}^{d+n}) \mid K \cap V = \emptyset\}.$$

Theorem 2.1. *The sets $\mathcal{N}_{K, \epsilon}(W)$ form the a system of neighbourhoods of a topology. Given W and K the set $\mathcal{N}_{K, \epsilon}(W)$ is open, for sufficiently small $\epsilon > 0$. With this topology, $\Psi_d(\mathbb{R}^{d+n})$ is a metrizable space.*

Proof. We give a proof of this lemma in section 4. □

We shall consider the subset $D_{d,n}^k \subset \Psi_d(\mathbb{R}^{d+n})$ of manifolds where the projection on the first k coordinates $f : W^d \rightarrow \mathbb{R}^k$ is proper, or said a little differently, let

$$D_{d,n}^k = \{W^d \in \Psi_d(\mathbb{R}^{d+n}) \mid W^d \subset \mathbb{R}^k \times \text{int}(I^{n+d-k})\},$$

where $\text{int}(I^N)$ is the open N -cube $(-1, 1)^N \subset \mathbb{R}^N$.

For a subset $S \subset \{1, \dots, k\}$ and $f : W \rightarrow \mathbb{R}^k$, let $f_S : W \rightarrow \mathbb{R}^S$ be the projection onto the coordinates given by S . If $\underline{a} \in \mathbb{R}^k$, let $\underline{a}_S = (a_i)_{i \in S} \in \mathbb{R}^S$ and define $Z(W)$ to be the subset of vectors $\underline{a} \in \mathbb{R}^k$ for which there exists an S such that f_S is *not* transversal

to \underline{a}_S . If $W \in D_{d,n}^k$ then $\underline{a} \notin Z(W)$ is the statement that $W^d \subset \mathbb{R}^k \times \text{int } I^{n+d-k}$ intersects all the affine subspaces

$$A(\underline{a}, S) = \{\underline{x} \in \mathbb{R}^{d+n} \mid x_i = \underline{a}_i \text{ for } i \in S\}$$

transversely. If we set

$$W(\underline{a}; S) = f_S^{-1}(\underline{a}_S) = W \cap A(\underline{a}, S)$$

then

$$W(\underline{a}; S) \cap W(\underline{a}; T) = W(\underline{a}; S \cup T)$$

with transverse intersection, provided $\underline{a} \notin Z(W)$. By Sard's theorem each $Z(W) \subset \mathbb{R}^k$ is closed and has measure 0. In order to apply the abstract theory of section 1 we consider

$$Z_{d,n}^k = \{(W, \underline{a}) \in D_{d,n}^k \times \mathbb{R}^k \mid \underline{a} \in Z(W)\}$$

We must show that the pair $(D_{d,n}^k, Z_{d,n}^k)$ is a critical pair, i.e.

Proposition 2.2. $Z_{d,n}^k$ is a closed subset of $D_{d,n}^k \times \mathbb{R}^k$.

Proof. We prove that the complement is open, so let $(W, \underline{a}) \notin Z_{d,n}^k$. It suffices to find a neighbourhood of (W, \underline{a}) in $D_{d,n}^k \times \mathbb{R}^k$ such that for each (V, \underline{b}) in this neighbourhood, $f : V \rightarrow \mathbb{R}^k$ is transverse to \underline{b} . Indeed, the argument below can be repeated for each $f_S : W \rightarrow \mathbb{R}^S$.

Since $f : W \rightarrow \mathbb{R}^k$ is proper, hence closed, the singular values is a closed subset of \mathbb{R}^k . Let $D_\epsilon \subset \mathbb{R}^k$ be an ϵ -disc around \underline{a} of regular values for f , and set $K_\epsilon = D_\epsilon(\underline{a}) \times I^{d+n-k} \subset \mathbb{R}^{d+n}$. Choose $\delta > 0$ so small that for $s \in \Gamma_{K_{\epsilon/2}, \delta}(s_0)$, where $s_0 : W \rightarrow NW$ denotes the zero section, one has

- (i) for $x \in K_{\epsilon/2} \cap W$, $s(x) \in K_\epsilon$,
- (ii) for $x \in K_\epsilon \cap W$, the differential $d(f \circ s)_x$ is surjective. Since f is the projection, $f \circ s = (s^1, \dots, s^k)$.

For $(V, \underline{b}) \in \mathcal{N}_{K_{\epsilon/2}, \delta}(W) \times D_{\epsilon/2}(\underline{a})$ we must show that $f : V \rightarrow \mathbb{R}^k$ is transverse to \underline{b} . Let $V = s(W)$ with $s \in \Gamma_{K_{\epsilon/2}, \delta}(s_0)$, and let $y = s(x) \in V$ be an element of $f^{-1}(\underline{b})$. By (i), $f \circ s(x) \in D_\epsilon(\underline{a})$, and by (ii),

$$T_x(W) \xrightarrow{ds_x} T_y(V) \xrightarrow{df_y} \mathbb{R}^k$$

is surjective. But ds_x is an isomorphism, so that $df_y : T_y V \rightarrow \mathbb{R}^k$ is surjective, and \underline{b} is a regular value. \square

Since $\psi_a(\mathbb{R}^{d+n})$ is metrizable by theorem 2.1 and $Z_{d,n}^k$ is closed by proposition 2.2 we can apply theorem 1.8 to get

Corollary 2.3. *The projection $p : |K_\bullet(D_{d,n}^k, Z_{d,n}^k)| \rightarrow D_{d,n}^k$ is a weak homotopy equivalence.* \square

S. Galatius in section 6 of [5] determined the homotopy type of $D_{d,n}^k$ by applying Gromov's theory of microflexible sheaves to the sheaf Ψ_d defined on open subsets $U \subset \mathbb{R}^{d+n}$,

$$\Psi_d(U) = \{W^d \subset U \mid \partial W^d = \emptyset, W^d \text{ a closed subset}\}.$$

This is a microflexible sheaf in the terminology of [7], see [?] for details, and the theory shows that “scanning” defines a homotopy equivalence

$$D_{d,n}^k \simeq \text{Map}((I^{d+n-k}, \partial I^{d+n-k}) \times \mathbb{R}^k, (\Psi_d(\mathbb{R}^{d+n}), \emptyset))$$

(cf. section 4.2 of [5]).

Let $U_{d,n}^\perp$ be the n -dimensional bundle over the Grassmannian $G(d, n)$ of d -dimensional linear subspaces of \mathbb{R}^{d+n} , consisting of pairs $(V, v) \in G(d, n) \times \mathbb{R}^{d+n}$ with $v \perp V$. There is an obvious map

$$q : U_{d,n}^\perp \rightarrow \Psi_d(\mathbb{R}^{d+n}), \quad q(V, v) = V - v,$$

which extends to a map of the Thom space $\text{Th}(U_{d,n}^\perp)$ into $\Psi_d(\mathbb{R}^{d+n})$ since $V - v$ leaves every compact subset as $v \rightarrow \infty$, and $\text{Th}(U_{d,n}^\perp)$ is the one point compactification of $U_{d,n}^\perp$. Lemma 6.1 of [5] shows that

$$q : \text{Th}(U_{d,n}^\perp) \rightarrow \Psi_d(\mathbb{R}^{d+n})$$

is a homotopy equivalence. Combined with corollary 2.3 we get

Theorem 2.4. $|K_\bullet(D_{d,n}^k, Z_{d,n}^k)| \simeq \Omega^{d+n-k} \text{Th}(U_{d,n}^\perp)$.

2.2 The k -category of manifolds in a cube.

We begin with a smooth submanifold

$$W_\epsilon^d \subset (-\epsilon, 1 + \epsilon)^k \times \text{int}(I^{d+n-k})$$

which is a closed subset of $(-\epsilon, 1 + \epsilon)^k \times \mathbb{R}^{d+n-k}$ and intersects $[0, 1]^k \times \text{int}(I^{d+n-k})$ orthogonally in a compact manifold with corners. More specifically, let $\underline{v} = (v^1, \dots, v^k)$ be a vertex of the k -dimensional cube $[0, 1]^k$ and let $S \subset \{1, \dots, k\}$ be any subset. For $\epsilon > 0$, define

$$\begin{aligned} A_S(\underline{v}, \epsilon) &= \{\underline{x} \in \mathbb{R}^{d+n} \mid v^i - \epsilon < x^i < v^i + \epsilon, i \in S\}, \\ A_S(\underline{v}) &= \{\underline{x} \in \mathbb{R}^{d+n} \mid x^i = v^i, i \in S\}. \end{aligned} \tag{5}$$

Notice that

$$A_S(\underline{v}, \epsilon) \cong A_S(\underline{v}) \times J_S(\underline{v}, \epsilon),$$

where $J_S(\underline{v}, \epsilon) = \prod_{i \in S} (v^i - \epsilon, v^i + \epsilon)$. We require W_ϵ^d to satisfy the following:

Condition 2.5.

- (i) W_ϵ is transverse to $A_S(\underline{v})$ for all vertices \underline{v} of $[0, 1]^k$,
- (ii) $W_\epsilon \cap A_S(\underline{v}, \epsilon) = (W_\epsilon \cap A_S(\underline{v})) \times J_S(\underline{v}, \epsilon)$,
- (iii) W_ϵ is a closed subset of $(-\epsilon, 1 + \epsilon)^k \times \mathbb{R}^{d+n-k}$.

The intersection $W = W_\epsilon \cap ([0, 1]^k \times \text{int}(I^{d+n-k}))$ is a compact manifold with corners. In the terminology of [11], it is a $\langle k \rangle$ -manifold embedded “neatly” in $[0, 1]^k \times \text{int}(I^{d+n-k})$ and equipped with a $\langle k \rangle$ -collar.

The size ϵ of this collar is not part of the structure. We tacitly form the colimit where ϵ tends to 0.

Given $\underline{a} \leq \underline{b}$ in \mathbb{R}^k , let

$$J(\underline{a}, \underline{b}) = \prod_{i=1}^k [a^i, b^i], \quad J_\epsilon(\underline{a}, \underline{b}) = \prod_{i=1}^k (a^i - \epsilon, b^i + \epsilon).$$

We consider d -dimensional submanifolds

$$W_\epsilon^d \subset J_\epsilon(\underline{a}, \underline{b}) \times \text{int}(I^{d+n-k}),$$

which satisfy the analogue of the three conditions in 2.5. Since W_ϵ is constant on the collars by condition (iii), it defines an element $\widehat{W}_\epsilon \in D_{d,n}^k$ with

$$W_\epsilon = \widehat{W}_\epsilon \cap (J_\epsilon(\underline{a}, \underline{b}) \times \text{int}(I^{d+n-k}))$$

upon extending the collars. We are more interested in the space $N_1 \mathcal{C}_{d,n}^k$ of all intersections

$$W = W_\epsilon \cap (J(\underline{a}, \underline{b}) \times \text{int}(I^{d+n-k})).$$

By the remarks above, $N_1 \mathcal{C}_{d,n}^k$ may be considered a subspace of $K_1(D_{d,n}^k, Z_{d,n}^k)$, cf. definition 1.2. There are k directions to compose elements of $N_1 \mathcal{C}_{d,n}^k$. This turns $\mathcal{C}_{d,n}^k$ into a strict k -tuple category. For $k = 1$ this is the category of embedded cobordisms, examined in [5], [6] for $n = \infty$. One can express the homotopy type of $N_1 \mathcal{C}_{d,n}^k$ in terms of classifying spaces. We sketch the result.

The cube $[0, 1]^k$ is a $\langle k \rangle$ -manifold with

$$\partial_i [0, 1]^k = [0, 1]^{i-1} \times \{0, 1\} \times [0, 1]^{k-i}.$$

Let W^d be a compact d -dimensional $\langle k \rangle$ -manifold with an ϵ -collar, cf. lemma 2.1.6 of [11] and let

$$\text{Emb}_\epsilon(W^d, [0, 1]^k \times \text{int}(I^{d+n-k}))$$

be the space of embeddings that maps the ϵ -collar of W^d to the ϵ -collar of $[0, 1]^k \times \text{int}(I^{d+n-k})$ in the obvious linear fashion. Let

$$\text{Emb}(W^d, [0, 1]^k \times \text{int}(I^{d+n-k})) = \lim_{\epsilon \rightarrow 0} \text{Emb}_\epsilon(W^d, [0, 1]^k \times \text{int}(I^{d+n-k})).$$

For small values of n , this space might be empty, namely if the given diffeomorphism type W does not embed in codimension n . The diffeomorphism group $\text{Diff}(W)$ of the collared $\langle k \rangle$ -manifold W acts freely on the embedding space, and the orbit

$$B_{d,n}^k(W) = \text{Emb}(W^d, [0, 1]^k \times \text{int}(I^{d+n-k})) / \text{Diff}(W)$$

is the set of collared $\langle k \rangle$ -submanifolds of $[0, 1]^k \times \text{int}(I^{d+n-k})$ diffeomorphic to W . For $n = \infty$,

$$B_{d,n}^k(W) \simeq B\text{Diff}(W)$$

by Whitney's embedding theorem. Let $C(k)$ be the space of all k -cubes with edges parallel to the axes,

$$C(k) = \{J(\underline{a}, \underline{b}) \in \mathbb{R}^k \mid \underline{a} < \underline{b}\}.$$

More generally, set

$$C(k-l) = \{J(\underline{a}, \underline{b}) \mid a^i = b^i \text{ for precisely } l \text{ indices}\}.$$

The subspace of non-degenerate morphisms of $\mathcal{C}_{d,n}^k$ is homeomorphic to $\coprod C(k) \times B_{d,n}^k(W^d)$ with W^d ranging over all $\langle k \rangle$ -manifolds of dimension d that embeds in $[0, 1]^k \times \text{int}(I^{d+n-k})$. If we intersect such an embedded manifold with one of the $(k-1)$ -dimensional faces we get a non-degenerate object of $\mathcal{C}_{d,n}^k$, alias a non-degenerate morphism of $\mathcal{C}_{d-1,n}^{k-1}$ etc. So we have

Proposition 2.6. *There is a homotopy equivalence*

$$N_1 \mathcal{C}_{d,n}^k \simeq \coprod_{l=0}^k \coprod_{W^{d-l}} C(k-l) \times B_{d-l,n}^{k-l}(W^{d-l}).$$

2.3 The homotopy type of $BC_{d,n}^k$

The multi-nerve of a strict k -tuple category is a k -dimensional simplicial space. The associated diagonal simplicial space is denoted $N_\bullet \mathcal{C}_{d,n}^k$. It is a subspace of $K_\bullet(D_{d,n}^k; Z_{d,n}^k)$.

Theorem 2.7. *The inclusion*

$$N_\bullet \mathcal{C}_{d,n}^k \rightarrow K_\bullet(D_{d,n}^k; Z_{d,n}^k)$$

induces a weak homotopy equivalence of realizations.

The proof will occupy the rest of this section, but before we embark on it, we list its obvious consequence

Theorem 2.8. *The weak homotopy type of $BC_{d,n}^k = |N_\bullet \mathcal{C}_{d,n}^k|$ is given by*

$$BC_{d,n}^k \simeq \Omega^{d+n-k} \text{Th}(U_{d,n}^\perp),$$

where $U_{d,n}^\perp$ is the n -dimensional canonical vector bundle over the Grassmannian $G(d, n)$ of d -planes in \mathbb{R}^{d+n} . In particular, we have the weak homotopy equivalence

$$\Omega BC_{d,n}^k \simeq BC_{d,n}^{k-1} \quad \text{for } 1 \leq k \leq d+n.$$

Remark 2.9. The above theorem works equally well for oriented manifolds letting $G(d, n)$ be the space of oriented d -planes, or more generally for the category $\mathcal{C}_{d,n}^k(\theta)$ of manifolds with a θ -structure in the sense of [6], section 5 or [15], section 2:

$$BC_{d,n}^k(\theta) \simeq \Omega^{d+n-k} \text{Th}(\theta^* U_{d,n}^\perp), \quad (6)$$

$$\Omega BC_{d,n}^k(\theta) \simeq BC_{d,n}^{k-1}(\theta) \quad \text{for } 1 \leq k \leq d+n. \quad (7)$$

The simplicial spaces $N_\bullet \mathcal{C}_{d,n}^k$ and $K_\bullet(D_{d,n}^k; Z_{d,n}^k)$ differ in two aspects. Elements of $N_\bullet \mathcal{C}_{d,n}^k$ intersect the facets of the k^2 cubes orthogonally in small collars whereas elements of $K_\bullet(D_{d,n}^k; Z_{d,n}^k)$ are merely transversal to the facets. The second difference is that the elements of $K_\bullet(D_{d,n}^k; Z_{d,n}^k)$ are supported on manifolds that are closed subsets of $\mathbb{R}^k \times \text{int}(I^{d+n-k})$ while elements of $N_\bullet \mathcal{C}_{d,n}^k$ are only subsets of an ϵ -collar of the union of cubes that is associated to the element. The inclusion of $N_\bullet \mathcal{C}_{d,n}^k$ into $K_\bullet(D_{d,n}^k; Z_{d,n}^k)$ is by extending the manifold in the ϵ -collar “linearly”.

The numbers d, n and k will be constant in the following, and we shall from now on drop the indices and simply write (D, Z) . We prove theorem 2.7 in two steps, first modifying elements of $K_1(D, Z)$ to have orthogonal intersection with the facets, and second making elements affine outside the ϵ -collar of the union of the k^2 cubes.

We say that an element $(W, \underline{a}) \in K_0(D, Z)$ has orthogonal corner structure (at \underline{a}) if for each $S \in \{1, \dots, k\}$

$$W \cap A_S(\underline{a}, \epsilon) = W_S(\underline{a}) \times J_S(\underline{a}, \epsilon). \quad (8)$$

for some $\epsilon > 0$. Here we use the notation of section 2.1, and in particular

$$J_S(\underline{a}, \epsilon) = \prod_{i \in S} (a^i - \epsilon, b^i + \epsilon).$$

Let $K_0^\perp(D, Z) \subset K_0(D, Z)$ be the subspace of elements with orthogonal corner structure at \underline{a} .

Lemma 2.10. *The inclusion $K_0^\perp(D, Z) \rightarrow K_0(D, Z)$ is a weak homotopy equivalence.*

Proof. Let $K_0(D, Z)(\underline{0})$ be the subspace of $K_0(D, Z)$ consisting of elements $(W, \underline{0})$. It is a deformation retract via the deformation $(W, \underline{a}) \mapsto (W - t\underline{a}, (1-t)\underline{a})$ as $0 \leq t \leq 1$. Similarly, $K_0^\perp(D, Z)(\underline{0})$ is a deformation retract of $K_0^\perp(D, Z)$, so it suffices to show that

$$K_0^\perp(D, Z)(\underline{0}) \rightarrow K_0(D, Z)(\underline{0})$$

is a weak homotopy equivalence.

Given $(W, \underline{0}) \in K_0(D, Z)(\underline{0})$, the projection $f_S : W \rightarrow \mathbb{R}^S$ is transversal to each point in $J_S(\underline{0}, \epsilon)$. Let $\lambda_\epsilon : \mathbb{R} \rightarrow \mathbb{R}$ be a fixed smooth function subject to the following requirements:

- (i) λ_ϵ is weakly increasing and proper,

(ii) $\lambda_\epsilon(x) = x$ for $x \leq -\epsilon$ and $x \geq \epsilon$,

(iii) $\lambda_\epsilon(x) = 0$ for $x \in (-\epsilon/2, \epsilon/2)$.

Let $\hat{\lambda}_\epsilon : \mathbb{R}^k \times \text{int}(I^{d+n-k}) \rightarrow \mathbb{R}^k \times \text{int}(I^{d+n-k})$ be the function that sends (x^1, \dots, x^k, y) to $(\lambda_\epsilon(x^1), \dots, \lambda_\epsilon(x^k), y)$. By the transversality assumption,

$$(\hat{\lambda}_\epsilon)^*(W) = \{(x, y) \in \mathbb{R}^k \times \text{int}(I^{d+n-k}) \mid \hat{\lambda}_\epsilon(\underline{x}, \underline{y}) \in W\}$$

is a submanifold, and one easily checks that

$$\begin{aligned} (\hat{\lambda}_\epsilon)^*(W)_S(\underline{0}) &= W_S(\underline{0}) \text{ , and} \\ (\hat{\lambda}_\epsilon)^*(W)_S \cap A(\underline{0}, \epsilon/2) &= W_S(\underline{0}) \times J_S(\underline{0}, \epsilon/2). \end{aligned}$$

Thus $(\hat{\lambda}_\epsilon)^*(W)$ has an orthogonal corner structure at $\underline{0}$. There is a path $(\hat{\lambda}_\epsilon^t)^*(W)$ from W to $(\hat{\lambda}_\epsilon)^*(W)$ given by the function

$$\hat{\lambda}_\epsilon^t(x) = (1-t)x + t\hat{\lambda}_\epsilon(x)$$

Note that the entire path $(\hat{\lambda}_\epsilon^t)^*(W)$ is in $K_0^\perp(D, Z)(\underline{0})$ when $W \in K_0^\perp(D, Z)(\underline{0})$

The number $\epsilon > 0$ depends on the given $W \in K_0(D, Z)(\underline{0})$, but can be kept constant in a neighbourhood of W ; this follows from Proposition 2.2 and its proof. Thus for each compact subset $C \subset K_0(D, Z)(\underline{0})$ there is an $\epsilon = \epsilon(C) > 0$ and a diagram

$$\begin{array}{ccc} C \times I & \xrightarrow{h_t} & K_0(D, Z)(\underline{0}) \\ \cup & & \cup \\ C \cap K_0^\perp(D, Z)(\underline{0}) & \xrightarrow{h_t^\perp} & K_0^\perp(D, Z)(\underline{0}) \end{array}$$

which for $t = 0$ is the inclusion diagram and such that $h_1 : C \rightarrow K_0^\perp(D, Z)(\underline{0})$. It follows that all relative homotopy groups $\pi_i(K_0(D, Z)(\underline{0}), K_0^\perp(D, Z)(\underline{0})) = 0$. \square

An element $(W, \underline{a}_0, \dots, \underline{a}_r) \in K_r(D, Z)$ gives rise to a subdivision of the cube $C(\underline{a}, \underline{b})$ into $(r-1)^k$ sub cubes. Define $K_r^\perp(D, Z)$ to be the subspace of $K_r(D, Z)$ of elements $(W, \underline{a}_0, \dots, \underline{a}_r)$ where $(W, \underline{v}) \in K_0^\perp(D, Z)$ for each sub cube vertex \underline{v} .

Corollary 2.11. *The inclusion $K_r^\perp(D, Z) \rightarrow K_r(D, Z)$ is a weak homotopy equivalence, $r \geq 0$.*

Proof. Apply the homotopy constructed in the proof of the the previous lemma simultaneously to $(W, \underline{v}) \in K_0(D, Z)$ for all vertices \underline{v} in the sub cubes. \square

Next we consider an embedding $N_\bullet \mathcal{C}_{d,n}^k \rightarrow K_\bullet^\perp(D_{d,n}^k, Z_{d,n}^k)$ where $N_\bullet \mathcal{C}_{d,n}^k$ is the diagonal simplicial space of the k -dimensional multi nerve. We first describe the image of the embedding.

For $\underline{a} \leq \underline{b}$, remember the notation

$$J(\underline{a}, \underline{b}) = \prod [a^i, b^i], \quad J_\epsilon(\underline{a}, \underline{b}) = \prod (a^i - \epsilon, b^i + \epsilon).$$

An element $(W, \underline{a}_0, \dots, \underline{a}_r) \in N_r \mathcal{C}_{d,n}^k$ is by definition the intersection of $W_\epsilon \subset J_\epsilon(\underline{a}_0, \underline{a}_r) \times \mathbb{R}^{d+n-k}$ with $J(\underline{a}_0, \underline{a}_r) \times \mathbb{R}^{d+n-k}$, where W_ϵ is a product in an ϵ -collar of the boundary of $J(\underline{a}_0, \underline{a}_r)$, cf. §(2.2). One defines $\hat{W}_\epsilon \subset \mathbb{R}^k \times \text{int}(I^{d+n-k})$ by extending the ϵ -collars, and obtain the embedding

$$N_r \mathcal{C}_{d,n}^k \hookrightarrow K_r^\perp(D_{d,n}^k, Z_{d,n}^k)$$

by sending $(W, \underline{a}_0, \dots, \underline{a}_r)$ to $(\hat{W}_\epsilon, \underline{a}_0, \dots, \underline{a}_r)$. There is a retraction by intersecting $(\hat{W}_\epsilon, \underline{a}_0, \dots, \underline{a}_r) \in K_\bullet^\perp(D_{d,n}^k, Z_{d,n}^k)$ with an ϵ -collar of the boundary of $J_\epsilon(\underline{a}_0, \underline{a}_r)$ for some ϵ . The elements of $N_r \mathcal{C}_{d,n}^k$ and $K_\bullet^\perp(D_{d,n}^k, Z_{d,n}^k)$ agree on $J_\epsilon(\underline{a}_0, \underline{a}_r) \times \mathbb{R}^{d+n-k}$ but may differ on the complement $(\mathbb{R}^k \setminus J_\epsilon(\underline{a}_0, \underline{a}_r)) \times \mathbb{R}^{d+n-k}$.

Theorem 2.12. *For each r , the inclusion*

$$N_r \mathcal{C}_{d,n}^k \hookrightarrow K_r^\perp(D_{d,n}^k, Z_{d,n}^k)$$

is a homotopy equivalence.

Proof. Given $(\hat{W}, \underline{a}_0, \dots, \underline{a}_r) \in K_r^\perp(D_{d,n}^k, Z_{d,n}^k)$ we must specify a curve from this element to $N_r \mathcal{C}_{d,n}^k$, independent of \hat{W} and depending continuously on $(\underline{a}_0, \dots, \underline{a}_r)$. The idea is to expand the outside collar of $J_\epsilon(\underline{a}_0, \underline{a}_r)$ without moving the collar of size $\epsilon/2$.

If $\underline{a}_0 < \underline{a}_r$, we can scale by an affine map to $\underline{a}_0 = (0, \dots, 0)$ and $\underline{a}_r = (1, \dots, 1)$, so that $J_\epsilon(\underline{a}_0, \underline{a}_r) = (-\epsilon, 1 + \epsilon)^k$. The degenerate situation $\underline{a}_0 = \underline{a}_r$ is similar but easier.

We introduce the following notation. For $0 < \epsilon \leq \mu$ let $D_{\mu,\epsilon}^\perp$ denote the space of submanifolds

$$W \subset (-\mu, 1 + \mu)^k \times \text{int}(I^{d+n-k})$$

which are closed as subsets of $(-\mu, 1 + \mu)^k \times \mathbb{R}^{d+n-k}$ and such that the restriction

$$\text{Res}_\epsilon^\mu(W) = W \cap (-\epsilon, 1 + \epsilon)^k \times \text{int}(I^{d+n-k})$$

satisfies the three conditions of (2.5).

We define the embedding by extending the outside collar

$$\phi_\epsilon : D_{\epsilon,\epsilon}^\perp \rightarrow D_{\infty,\epsilon}^\perp$$

as follows. Choose a standard diffeomorphism

$$\phi_\epsilon : (-\epsilon, 1 + \epsilon) \rightarrow \mathbb{R}.$$

The $\phi_\epsilon^k \times \text{id}$ is a diffeomorphism from $(-\epsilon, 1 + \epsilon)^k \times \text{int}(I^{d+n-k})$ to $\mathbb{R}^k \times \text{int}(I^{d+n-k})$ and

$$\hat{\phi}_\epsilon(W) := (\phi_\epsilon^k \times \text{id})(W)$$

We have

$$\text{Res}_\epsilon^\infty \circ \hat{\phi}_\epsilon \simeq \text{id} \text{ and } \hat{\phi}_\epsilon \circ \text{Res}_\epsilon^\infty \simeq \text{id} \quad (9)$$

The first equivalence is obvious. The second homotopy is given in the following way. For $t \geq \epsilon$, let $\rho_t : \mathbb{R} \rightarrow \mathbb{R}$ be the affine map with $\rho_t(-t) = -\epsilon$ and $\rho_t(1+t) = 1+\epsilon$. Set $\psi_t = \rho_t^{-1} \circ \phi_\epsilon \circ \rho_t$ and consider

$$\hat{\psi}_t : D_{t,\epsilon}^\perp \rightarrow D_{\infty,\epsilon}^\perp$$

Since ψ_t is constant on subintervals of $(-t, 1+t)$ that tends to $(-\infty, \infty)$ for $t \rightarrow \infty$, the composition

$$\hat{\psi}_t \circ \text{Res}_t^\infty : D_{\infty,\epsilon}^\perp \rightarrow D_{\infty,\epsilon}^\perp$$

has limit $\hat{\psi}_\infty = \text{id}$ at $t \rightarrow \infty$. Thus $W_t = \hat{\psi}_t \circ \text{Res}_t^\infty(W)$ is a curve from $\hat{\phi}_\epsilon \circ \text{Res}_\epsilon^\infty(W)$ to W at $t \in [\epsilon, \infty)$. This is the required homotopy in (9). \square

A map of simplicial spaces $X_\bullet \rightarrow Y_\bullet$ which is a degreewise homotopy equivalence induces a (weak) homotopy equivalence of topological realizations, so theorem 2.7 is a consequence of theorem 2.12.

It is sometimes more convenient to work with the simplicial space of discrete cut sets $K_\bullet^\delta(D_{d,n}^k, Z_{d,n}^k)$ rather than with $K_\bullet(D_{d,n}^k, Z_{d,n}^k)$ where the cut points can move continuously. We let $N_\bullet^\delta \mathcal{C}_{d,n}^k$ be the set $N_\bullet \mathcal{C}_{d,n}^k$, but re-topologized as a subset of $K_\bullet^\delta(D_{d,n}^k, Z_{d,n}^k)$. This gives the diagram of simplicial spaces

$$\begin{array}{ccc} N_\bullet^\delta \mathcal{C}_{d,n}^k & \longrightarrow & K_\bullet^\delta(D_{d,n}^k, Z_{d,n}^k) \\ \downarrow & & \downarrow \\ N_\bullet \mathcal{C}_{d,n}^k & \longrightarrow & K_\bullet(D_{d,n}^k, Z_{d,n}^k) \end{array} \quad (10)$$

where the horizontal arrows are inclusions. In the proof above of theorem 2.12, we did not move the cut points, so the same argument gives

Addendum 2.13. The map $N_\bullet^\delta \mathcal{C}_{d,n}^k \rightarrow K_\bullet^\delta(D_{d,n}^k, Z_{d,n}^k)$ induces a weak homotopy equivalence.

The right hand vertical map in (10) is a weak homotopy equivalence by theorem 2.12, so (10) is a diagram of weak homotopy equivalences.

3 Simplicial spaces

The purpose of this section is to prove some facts about the realizations of simplicial spaces that we need for the proof of theorem 1.7. We construct a regular neighbourhood of a degreewise open subset, and apply this to give a criterion that ensures that a map with contractible fibres is a homotopy equivalence.

3.1 The second derived neighbourhood of simplicial spaces

This section contains a version of the regular neighbourhood theorem for a pair of simplicial spaces. Suppose that $Y_\bullet \subset X_\bullet$ is a simplicial space with a simplicial subspace, and assume for convenience that the spaces are degreewise compactly generated ([22]). This is the case for example if X_\bullet consists of metrizable spaces. Suppose that in each degree the inclusion is the inclusion of an open subspace. As the special case of simplicial sets shows, we cannot expect that the induced map of realizations is the inclusion of an open subset. For a discussion of this, see [14].

Let the open star $\text{St}(X_\bullet, Y_\bullet)$ be the union of all open simplices t in $|X_\bullet|$ such that at least one vertex of t is contained in Y_0 . We consider the vertex maps $v_i: X_n \rightarrow X_0$, ($0 \leq i \leq n$) that maps a simplex to its i^{th} vertex (induced by the inclusion $[0] \ni 0 \mapsto i \in [n]$).

Lemma 3.1. *Assume that Y_n is an open subset of X_n . Then the open star $\text{St}(X_\bullet, Y_\bullet)$ is an open subset of $|X_\bullet|$.*

Proof. Let $\phi_n: X_n \times \Delta^n \rightarrow |X_\bullet|$ be the characteristic map. By the definition of the topology of the realization, it is enough to show that for every n , the set $\phi_n^{-1}(\text{St}(X_\bullet, Y_\bullet))$ is open.

Let $Y'_n = \{x \in X_n \mid \underline{v}_i(x) \in Y_0 \text{ for some } i\}$. Since Y_0 is open in X_0 and each v_i is continuous this is an open subset of X_n . If $\alpha: [k] \rightarrow [n]$ is a morphism in the simplicial category, then $(\alpha^*)^{-1}(Y'_k) \subset Y'_n$. By definition,

$$\text{St}(X_\bullet, Y_\bullet) = \bigcup_k \phi_k(Y'_k \times \text{int}(\Delta^k)).$$

A point in $|X_\bullet|$ is uniquely represented by some $(y, t) \in X_n \times \text{int}(\Delta^n)$, so this union is actually a disjoint union. Moreover, if $(x, s) \in X_k \times \Delta^k$ is any other representative of the same point, there is an injective morphism $\alpha: [k] \rightarrow [n]$ such that $\alpha_*(t) = s$ and $\alpha^*(x) = y$ ([17], lemma 14.2).

Given a point $(y, t) \in Y'_k \times \text{int}(\Delta^k)$ we have to show the following property: If $(x, s) \in X_n \times \Delta^n$ represents the same point as (y, t) in $|X_\bullet|$, then (x, s) is an inner point of $\phi_n^{-1}(\text{St}(X_\bullet, Y_\bullet))$. Let $\alpha: [k] \rightarrow [n]$ be as above. If $k = n$, the openness follows because Y'_n is an open set, so we can assume that $k < n$, and $s \in \partial\Delta^n$.

We claim that there is an open neighbourhood $U \subset \Delta^n$ of s , such that $(\alpha^*)^{-1}Y'_k \times U$ is an open set contained in $\phi_n^{-1}(\text{St}(X_\bullet, Y_\bullet))$. By induction on n , we can assume that there is an open neighbourhood V of s in $\partial\Delta^n$, so that

$$(\alpha^*)^{-1}Y'_k \times V \subset \phi_n^{-1}(\text{St}(X_\bullet, Y_\bullet)) \cap X_n \times \partial\Delta^n.$$

On the other hand, $(\alpha^*)^{-1}Y'_k \times \text{int}(\Delta^n) \subset Y'_n \times \text{int}(\Delta^n) \subset \phi_n^{-1}(\text{St}(X_\bullet, Y_\bullet)) \cap X_n \times \text{int}(\Delta^n)$, so that we can find the wanted neighbourhood U by choosing it as an arbitrary open neighbourhood of s in Δ^n such that $U \cap \partial\Delta^n \subset V$. \square

The open star construction gets better after subdivision. We remember that the subdivision $\text{Sd}X_\bullet$ is the nerve of the topological category of simplices of X_\bullet ; it has

objects $([n], x)$ with $x \in X_n$ and a morphism from $([n], x)$ to $([m], y)$ is a morphism $\alpha : [n] \rightarrow [m]$ with $\alpha^*(y) = x$.

Let $x \in \text{Sd}X_n$ be $x = ([N_0] \rightarrow \cdots \rightarrow [N_n], z \in X_{N_n})$. Its i^{th} vertex is $v_i(x) = ([N_i], \beta^*(z))$, with $\beta : [N_i] \rightarrow \cdots \rightarrow [N_n]$. If $v_n(x) = ([N_n], z) \in \text{Sd}Y_0$, i.e. if $z \in Y_{N_n}$ then $x \in \text{Sd}Y_{N_n}$. The pair $(Z_\bullet, T_\bullet) = (\text{Sd}X_\bullet, \text{Sd}Y_\bullet)$ thus has the following

Property I: A point $z \in Z_n$ is contained in T_n if and only if $v_n(z) \in T_0$.

Remark 3.2. Suppose that the pair Z_\bullet, T_\bullet has property I, and that $z \in (\text{Sd}Z)_n$ has the property that some vertex $v_i(z) \in (\text{Sd}T)_0$. Then the last vertex $v_n(z) \in T_0 \subset Z_0$.

Lemma 3.3. *Let X_\bullet be a degreewise compactly generated simplicial space. Suppose that for every i the inclusion $Y_i \subset X_i$ is an open embedding, and that the pair (X_\bullet, Y_\bullet) has Property I. Then, the inclusion $|Y_\bullet| \subset \text{St}(X_\bullet, Y_\bullet)$ is a homotopy equivalence.*

Proof. We consider two natural maps associated to a subdivision of a simplicial set X_\bullet . The first one is the subdivision map

$$s_X : |\text{Sd}X_\bullet| \rightarrow |X_\bullet|.$$

This is the unique map that is affine on simplices, and sends a vertex $a \in X_n \subset \text{Sd}X_0$ to the barycenter of the simplex represented by a in $|X_\bullet|$. The subdivision map is not a simplicial map, but it is a homeomorphism if X_\bullet is assumed to be degreewise compactly generated. To see that s_X^{-1} is continuous, consider for each N the diagram

$$\begin{array}{ccc} \coprod_{[N_0] \rightarrow \cdots \rightarrow [N_{r-1}] \rightarrow [N]} \Delta^r \times X_N & \xrightarrow{\phi_{\text{Sd}X}} & |\text{Sd}X_\bullet| \\ \downarrow f & \searrow s_X^{-1} \phi_X & \downarrow s_X \\ \Delta^N \times X_N & \xrightarrow{\phi_X} & |X_\bullet|. \end{array}$$

The vertical map $f : \Delta^r \times X_N \rightarrow \Delta^N \times X_N$, associated with $[N_0] \rightarrow [N_1] \rightarrow \cdots \rightarrow [N_r]$, ($N_r = N$), is the product of the identity on X_N and the following map $\Delta^r \rightarrow \Delta^N$: each $[N_i] \rightarrow [N_r]$ induces a simplicial $\Delta^{N_i} \xrightarrow{l_i} \Delta^{N_r}$; let b_i be the barycenter of the image $l_i(\Delta^{N_i})$. Then $f : \Delta^r \rightarrow \Delta^N$ is the affine map that takes the i^{th} vertex in Δ^r to b_i .

The inverse of s_X is continuous if and only if $(s_X)^{-1} \phi_X$ is continuous. But this follows because if X_N is compactly generated, then f is an identification map (a surjection map, where the target has the quotient topology), since it is the product of a compact identification map with the identity on a compactly generated space ([22]).

The second map we consider is a “first vertex map”. It is a simplicial map given in the following fashion. Let

$$[N_0] \xrightarrow{\alpha_0} [N_1] \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{n-1}} [N_n]$$

be an n -simplex in the subdivision. The sequence determines a map in the simplicial category $\alpha : [n] \rightarrow [N_n]$ by defining $\alpha(i) \in [N_n]$ to be the image under the iterated maps

of the first vertex $0 \in [N_{n-i}]$, that is $\alpha(i) = \alpha_{n-1} \circ \alpha_{n-2} \circ \cdots \circ \alpha_{n-i}(0)$. The first vertex map is the continuous simplicial map

$$L_X: \text{Sd}X_\bullet \rightarrow X_\bullet, \quad L_X(x, \alpha_0, \alpha_1, \dots, \alpha_{n-1}) = \alpha^*(x).$$

Let $z = (x, \alpha_0, \alpha_1, \dots, \alpha_{n-1}) \in \text{Sd}X_\bullet$. If $v_i(z) \in \text{Sd}Y_\bullet$ for some i , then $L_X(z) \in Y_n$. The condition means that $(\alpha_{n-1} \circ \cdots \circ \alpha_i)^*(x) \in Y_{N_i}$. Since Y_\bullet is a simplicial subset, it follows that $v_n(L_X(x)) = (\alpha_{n-1} \circ \cdots \circ \alpha_0 \circ v_0)^*(x) \in Y_0$. From property I it follows that $L_X(x) \in Y_n$. We conclude that $|L_X|(\text{St}(\text{Sd}X_\bullet, \text{Sd}Y_\bullet)) \subset |\text{Sd}Y_\bullet|$.

The first vertex map does not induce a homeomorphism, but the two maps $|L|, s_X: |\text{Sd}X_\bullet| \rightarrow |X_\bullet|$ are homotopic. Indeed, if $(z, t) \in \text{Sd}X_n \times \Delta^n$ represents a point $\phi_n(z, t) \in |\text{Sd}X_\bullet|$, then z is given by $x \in X_{N_n}$ and a sequence of morphisms $\{\alpha_i\}$ in the simplicial category. We compute

$$\begin{aligned} |L_X|(\phi_n(z, t)) &= \phi_{N_n}(x, \alpha_* t) \\ s_X(\phi_n(z, t)) &= \phi_{N_n}(x, A(t)). \end{aligned}$$

where $A: \Delta^n \rightarrow \Delta^{N_n}$ is an affine map, depending on the sequence of morphisms $\{\alpha_i\}$. A homotopy H_s from $|L_X|$ to s_X is given by

$$H_n(\phi_{N_n}(z, t)) = \phi_{N_n}(x, ((1-s)\alpha_*(t) + sA(t))).$$

We can restrict the two maps from $|\text{Sd}X_\bullet|$ to $\text{St}(\text{Sd}(X)_\bullet, \text{Sd}(Y)_\bullet)$. When we do this for L_X we get a commutative diagram

$$\begin{array}{ccccc} |Y_\bullet| & \xrightarrow{s_Y^{-1}} & |\text{Sd}Y_\bullet| & \xrightarrow{i} & \text{St}(\text{Sd}X_\bullet, \text{Sd}Y_\bullet) & \subseteq & |\text{Sd}X_\bullet| \\ & & \downarrow L_Y & & \downarrow L_X & & \downarrow L_X \\ & & |Y_\bullet| & \xlongequal{\quad} & |Y_\bullet| & \xrightarrow{\quad} & |X_\bullet| \end{array}$$

and it suffices to show that the map $i \circ s_Y^{-1}: |Y_\bullet| \rightarrow \text{St}(\text{Sd}X_\bullet, \text{Sd}Y_\bullet)$ has the homotopy inverse $L_X: \text{St}(\text{Sd}X_\bullet, \text{Sd}Y_\bullet) \rightarrow |Y_\bullet|$.

The first vertex map L_Y is homotopic to s_Y , so that the composition

$$|Y_\bullet| \xrightarrow{s_Y^{-1}} |\text{Sd}Y_\bullet| \xrightarrow{i} \text{St}(\text{Sd}X_\bullet, \text{Sd}Y_\bullet) \xrightarrow{L_X} |Y_\bullet|$$

is homotopic to the identity. Composing the other way, we obtain a commutative diagram

$$\begin{array}{ccccccc} \text{St}(\text{Sd}X_\bullet, \text{Sd}Y_\bullet) & \xrightarrow{L_X} & |Y_\bullet| & \xrightarrow{s_Y^{-1}} & |\text{Sd}Y_\bullet| & \xrightarrow{i} & \text{St}(\text{Sd}X_\bullet, \text{Sd}Y_\bullet) \\ \bigcap & & \bigcap & & \bigcap & & \bigcap \\ |\text{Sd}X_\bullet| & \xrightarrow{L_X} & |X_\bullet| & \xrightarrow{s_X^{-1}} & |\text{Sd}X_\bullet| & \xlongequal{\quad} & |\text{Sd}X_\bullet|. \end{array}$$

The composite of the lower row $s_X^{-1}L_X$ is homotopic to the identity by the homotopy $s_X^{-1}H_s$. To finish the proof of the lemma, we have to argue that this homotopy preserves

the subspace $\text{St}(\text{Sd}X_\bullet, \text{Sd}Y_\bullet)$. This is equivalent to the statement that if $(z, t) \in \text{Sd}X_n \times \text{int}(\Delta^n)$ represents a point $\phi_n(z, t) \in \text{St}(\text{Sd}X_\bullet, \text{Sd}Y_\bullet) \subset |X_\bullet|$, then

$$H_s(\phi_n(z, t)) \in s_X \text{St}(\text{Sd}X_\bullet, \text{Sd}Y_\bullet) \subset |X_\bullet|.$$

We examine $\phi_n^{-1}(|Y_\bullet|)$ and $\phi_n^{-1}(\text{St}(\text{Sd}X_\bullet, \text{Sd}Y_\bullet))$. We are assuming property I, so for a fixed $x \in X_N$, there will be some k , $-1 \leq k \leq n$ such that $v_i(x) \in \text{Sd}Y_0$ for $i \leq k$, and $v_i(x) \notin \text{Sd}Y_0$ for $i > k$. This means that $\{t \in \Delta^n \mid (x, t) \in \phi_n^{-1}(|Y_\bullet|)\}$ will be the convex span of the vertices $\{v_i \mid 0 \leq i \leq k\}$, which is either the empty set (in case $k = -1$), or a sub-simplex of Δ^n .

It follows that $(x, t) \in \phi_N^{-1}(\text{St}(\text{Sd}X_\bullet, \text{Sd}Y_\bullet))$ if and only if there is an $i \leq k$ such that $t_i > t_j$ for $k < j \leq n$.

Let $(z, t) \in (\text{Sd}X_n \times \text{int}(\Delta^n))$ represent a point in $\text{St}(\text{Sd}X_\bullet, \text{Sd}Y_\bullet) \subset |X_\bullet|$. If $z \in \text{Sd}X_r$ is represented by $[N_0] \xrightarrow{\alpha_0} \dots \xrightarrow{\alpha_{n-1}} [N_n]$ together with $x \in X_{N_n}$, then the image of (z, t) under the homotopy is represented by a line segment in $\{x\} \times \Delta^{N_n} \subset X_{N_n} \times \Delta^{N_n}$ which connects $(x, \alpha_*(t))$ to $(x, A(t))$, where $s_X(\phi_n(z, t)) = (x, A(t))$. By definition, $(x, A(t)) \in s_X \text{St}(\text{Sd}X_\bullet, \text{Sd}Y_\bullet)$, so there exists some $i \leq k$ such that $(\alpha_*(t))_i > (\alpha_*(t))_j$ for all $j > k$. We also know that $L_X(\phi_n(z, t)) \in |Y_\bullet|$, so $A(t)_j = 0$ for $j < k$. It follows that any convex combination $u = (1-s)\alpha_*(t) + sA(t)$ with $s < 1$ also satisfies that $u_i > u_j$ for all $j > k$, so that

$$H_s(\phi_n(z, t)) = \phi_{N_n}(x, u) \in s_X \text{St}(\text{Sd}X_\bullet, \text{Sd}Y_\bullet)$$

for $0 \leq s \leq 1$. □

Let $\text{Sd}^2 X_\bullet = \text{Sd}(\text{Sd}X_\bullet)$ be the second barycentric subdivision. The following theorem is an immediate consequence of our work.

Theorem 3.4. $|\text{St}(\text{Sd}^2 X_\bullet, \text{Sd}^2 Y_\bullet)|$ is an open set in $|\text{Sd}^2 X_\bullet|$. It contains $|(\text{Sd}^2 Y)_\bullet|$, and the inclusion of this subspace in $|\text{St}(\text{Sd}^2 X_\bullet, \text{Sd}^2 Y_\bullet)|$ is a weak homotopy equivalence.

Proof. By lemma 3.1 the star is an open subset. We have checked that $\text{Sd}X_\bullet$ has property I, so the theorem follows from lemma 3.3. □

We conclude the section with an application of theorem 3.4.

Let $f : X_\bullet \rightarrow Y_\bullet$ be a map of simplicial, degreewise metrizable spaces. Suppose that $X_\bullet^\alpha \subset X_\bullet$ and $Y_\bullet^\alpha \subset Y_\bullet$ are families of degreewise open subspaces, indexed by the same set A such that $f(|X_\bullet^\alpha|) \subset |Y_\bullet^\alpha|$ for $\alpha \in A$ and $X_\bullet = \cup X_\bullet^\alpha$, $Y_\bullet = \cup Y_\bullet^\alpha$.

Theorem 3.5. Suppose that for each finite subset $I \subset A$ the restriction

$$f : \bigcup_{\alpha \in I} |X_\bullet^\alpha| \subset \bigcup_{\alpha \in I} |Y_\bullet^\alpha|$$

is a weak homotopy equivalence. Then $f : |X_\bullet| \rightarrow |Y_\bullet|$ is a weak homotopy equivalence.

Proof. We do a double subdivision, and prove that $f : |\mathrm{Sd}^2(X_\bullet)| \rightarrow |\mathrm{Sd}^2(Y_\bullet)|$ is a weak homotopy equivalence.

By the assumption and by theorem 3.4 $\{\mathrm{St}(\mathrm{Sd}^2 Y_\bullet, \mathrm{Sd}^2 Y_\bullet^\alpha)\}_\alpha$ is an open cover of $|\mathrm{Sd}^2 Y_\bullet|$.

Let $g : A \rightarrow |\mathrm{Sd}^2 Y_\bullet|$ be a map from a finite CW complex. We show that this map factors up to homotopy over f . By compactness, it's image is contained in a finite union

$$\bigcup_i \{\mathrm{St}(\mathrm{Sd}^2 Y_\bullet, \mathrm{Sd}^2 Y_\bullet^{\alpha_i})\}.$$

The operation of forming the star is compatible with taking union of simplicial subspaces, so this is the same as the subspace

$$\mathrm{St}(\mathrm{Sd}^2 Y_\bullet, \mathrm{Sd}^2(\cup_i Y_\bullet^{\alpha_i})).$$

According to theorem 3.4 this space is weakly homotopy equivalent to $|\mathrm{Sd}(\mathrm{Sd}(\cup_i Y_\bullet^{\alpha_i}))|$, so f factors up to homotopy over the inclusion of this subspace. But then, by assumption, f factors up to homotopy over $|\mathrm{Sd}(\mathrm{Sd}(\cup_i X_\bullet^{\alpha_i}))|$. It follows that the map of homotopy classes

$$f_* : [A, |X_\bullet|] \rightarrow [A, |Y_\bullet|].$$

is surjective. A relative argument proves that the map is injective. \square

3.2 A lemma in homotopy theory.

Suppose that $f : X \rightarrow Y$ is a map with the property that point inverses are contractible. In many cases, this implies that f is a homotopy equivalence. For instance, if f is proper and X and Y satisfy very general conditions, this is proved by Smale ([21]). However, the statement is not true without topological assumptions on f . A simple counter example is given by $X = [0, 1) \cup \{2\} \subset \mathbb{R}$, $Y = [0, 1]$, $f(2) = 1$ and $f(x) = x$ for $x \neq 1$. We want to give a set of conditions that ensures homotopy equivalence in some non-proper cases. Suppose that X is a finite polyhedron (that is, the realization of a finite simplicial set), Y a topological space and $U \subset X \times Y$ an *open* set. Let $\pi_X : X \times Y \rightarrow X$, $\pi_Y : X \times Y \rightarrow Y$ be the projections.

Lemma 3.6. *Assume for each $x \in X$ that $\pi_Y : U \cap \pi_X^{-1}\{x\} \rightarrow Y$ is a weak homotopy equivalence. Then the inclusion of U in $X \times Y$ is a weak homotopy equivalence.*

Before we embark on the proof, we note that in combination with theorem 3.4 the lemma leads to the following conclusion:

Theorem 3.7. *Let X be a finite polyhedron and N_\bullet a degreewise compactly generated simplicial space. Let K_\bullet be a simplicial space and $K_\bullet \subset X \times N_\bullet$ a degreewise open subspace. Let $\pi_X : K_\bullet \rightarrow X$ be the projection. Assume for each $x \in X$ that the fiber $|\pi_X^{-1}(x)| \subset |K_\bullet|$ is contractible. Then $\pi_X : |K_\bullet| \rightarrow X$ is a homotopy equivalence.*

Proof. Let $\text{St}(\text{Sd}^2(X \times N_\bullet, \text{Sd}^2 K_\bullet) \subset X \times |N_\bullet|$ be the open star in the second subdivision, and consider the diagram

$$\begin{array}{ccccc} |K_\bullet| & \xrightarrow{\simeq} & \text{St}(\text{Sd}^2(X \times N_\bullet, \text{Sd}^2 K_\bullet)) & \longrightarrow & X \times |N_\bullet| \\ & \searrow \pi_X & \downarrow \pi_{\text{St}} & \swarrow \text{pr}_X & \\ & & X & & \end{array}$$

According to theorem 3.4 the first horizontal map is a homotopy equivalence, and since the star is open it suffices to check that $\pi_X^{-1}(x)$ is contractible. But $\pi_{\text{St}}^{-1}(x)$ is the star of the second derived neighbourhood of $\pi_X^{-1}(x)$ in the simplicial space $\{x\} \times |N_\bullet|$, and hence contractible by the assumption. \square

Remark 3.8. We point out that theorem 3.7 together with theorem 3.4 proves theorem 1.8, since $N_\bullet = N_\bullet(\mathbb{R}^k)$ is contractible so that $|K_\bullet(X, Z)| \subset |X \times N_\bullet(\mathbb{R}^k)|$ satisfies the assumptions of theorem 3.7.

The remainder of this section provides a proof of lemma 4.12. Given a pair of finite polyhedra $P \subset Q$ and a commutative diagram

$$\begin{array}{ccc} P & \subset & Q \\ \downarrow & & \downarrow f \\ U & \subset & X \times Y \end{array}$$

we must show that f is homotopic to a map with image in U , by a homotopy that maps P to U at each stage. To begin with, we may (and will) assume that the first coordinates $f_X = \pi_X \circ f : Q \rightarrow X$ and $f_X|_P$ are simplicial maps. This follows from simplicial approximation. We will further assume that for each simplex Δ of P we have that $f(\Delta) \subset V \times W \subset U$ for open sets $U \subset X$ and $W \subset Y$. (This might require that we further subdivide P and Q), Below we shall use the terminology that a subset $A \subset U$ is neatly contained in U if $\pi_X(A) \times \pi_Y(A) \subset U$.

Step 1. We claim that there is a subdivision of X such that for each simplex Δ_α in the subdivision we have the following property of the pair $Q_\alpha = f_X^{-1}(\Delta_\alpha)$, $P_\alpha = Q_\alpha \cap P$: There is a homotopy $H_\alpha : Q_\alpha \times I \rightarrow X \times Y$ of $f_\alpha = f|_{Q_\alpha}$ satisfying

- (i) $\pi_X \circ H_\alpha(q, t) = \pi_X \circ f(q)$ for $q \in Q_\alpha$.
- (ii) $H_\alpha(p, t) = f(p)$ for $p \in P_\alpha$.
- (iii) $H_\alpha(Q_\alpha, 1) \subset \Delta_\alpha \times W_\alpha \subset U$ for some open $W_\alpha \subset Y$.

Note that (i) is the statement that the homotopy H_α is effectively a homotopy of $\pi_Y \circ f_\alpha$. We now proceed to prove the claim. For $x \in X$, let $Q_x = f_X^{-1}(\{x\} \times Y)$ and $P_x = P \cap Q_x$. Consider the diagram

$$\begin{array}{ccc} P_x & \subset & Q_x \\ \downarrow & & \downarrow f \\ U_x & \subset & \{x\} \times Y. \end{array}$$

By assumption, we can find a homotopy $H_x: Q_x \times I \rightarrow \{x\} \times Y$, constant on P_x , from the restriction $f|_{Q_x}$ to a map with target inside U_x . We can extend the homotopy by a constant map on P , to obtain a homotopy from the restriction $f: Q_x \cup P \rightarrow X \times Y$ to a map into U . By construction, this homotopy is fibrewise over X . Since Q_x and P are sub-polyhedra of Q , we can extend this homotopy to a homotopy $H_x: Q \times I \rightarrow X \times Y$ with $\pi_X \circ H_x(q, t) = f_X(q)$.

Let $h(q) = H_x(q, 1)$. For every point $q \in h(Q_x)$ there is a neighbourhood V_q of $q \in X$ and an open set $W_q \in Y$ such that $h(q) \subset V_q \times W_q \subset U$. By compactness we can cover Q_x by finitely many open sets $h^{-1}(V_{q_i} \times W_{q_i})$. Put $V'_x = \cap_i V_{q_i}$ and $W_x = \cup_i W_{q_i}$, so that $Q_x \subset h^{-1}(V'_x \times W_x) \subset U$. The closed set $Q \setminus h^{-1}(V'_x \times W_x)$ is compact, so $A_x = \pi_X h(Q \setminus h^{-1}(V'_x \times W_x))$ is a closed set in X , not containing x .

Let $V_x = V'_x \cap (X \setminus A_x)$. For any $q \in (\pi_X f)^{-1}(V_x)$, we have that $h(q) \in V_x \times W_x \subset U$, so that $h(\pi_X f)^{-1}(V_x) \subset V_x \times W_x \subset U$. It follows that $h(\pi_X f)^{-1}(V_x)$ is neatly contained in U .

Using the compactness of X we can find a finite covering by such open sets V_x . After some additional subdividing each simplex Δ_α of X will be contained inside one of the sets V_x . Let H_α be the restriction of H_x to $Q_\alpha = f_X^{-1}(\Delta_\alpha)$. This completes the proof of step 1.

In step 1 we subdivided X to obtain that for each simplex Δ_α (in the subdivision) we have a homotopy $H_\alpha: Q_\alpha \times I \rightarrow \Delta_\alpha \times Y$ from $f|_{Q_\alpha}$ to a map $h_\alpha: Q_\alpha \rightarrow U \cap (\Delta_\alpha \times Y)$. We next make induction over the skeletons of X . For the induction step, assume that X is n -dimensional and let $X^{n-1} \subset X$ be the $(n-1)$ -skeleton. The induction hypothesis is that a diagram

$$\begin{array}{ccc} P_{n-1} & \subset & Q_{n-1} \\ \downarrow & & \downarrow f_{n-1} \\ U_{n-1} & \subset & X^{n-1} \times Y, \end{array} \quad U_{n-1} = U \cap (X^{n-1} \times Y)$$

with Q_{n-1}, P_{n-1} a pair of polyhedrons permits a homotopy $F_{n-1}: Q_{n-1} \times I \rightarrow X^{n-1} \times Y$ from f_{n-1} to a map that sends Q_{n-1} into U ; the homotopy is relative to P_{n-1} in the sense that $F_{n-1}(p, t) = f_{n-1}(p)$ for $p \in P_{n-1}$.

Let $\{\Delta_\alpha | \alpha \in A\}$ be the n -simplices of X so that $X = X^{n-1} \cup \bigcup_{\alpha \in A} \Delta_\alpha$.

Step 2. For each n -simplex Δ_α we have the homotopy $H_\alpha: Q_\alpha \times I \rightarrow \Delta_\alpha \times Y$ constructed in step 1. We first modify H_α as follows.

Choose for each $\alpha \in A$ a small affine subsimplex $\Delta_\alpha^0 \subset \Delta_\alpha$ around the barycenter and pick a function

$$s_\alpha: \Delta_\alpha \times I \rightarrow I$$

with $s_\alpha(x, t) = 0$ if $x \in \partial\Delta_\alpha$, $s_\alpha(x, t) = t$ if $x \in \Delta_\alpha^0$. Define $G_\alpha: Q \times I \rightarrow X \times Y$ to be

$$G_\alpha(q, t) = \begin{cases} H_\alpha(q, s_\alpha(f_X(q), t)), & q \in Q_\alpha \\ f(q) & \text{otherwise.} \end{cases}$$

where $Q_\alpha = f^{-1}(\Delta_\alpha \times Y)$. For $q \in Q_\alpha^0 = f^{-1}(\Delta_\alpha^0 \times Y)$, $G_\alpha(q, t) = H_\alpha(q, t)$ and it follows from (iii) above that $H_\alpha(Q_\alpha^0, 1) \subset \Delta_\alpha \times W_\alpha$ for some $W_\alpha \subset Y$.

The homotopies G_α , $\alpha \in A$ glue together to define a homotopy

$$G : Q \times I \rightarrow X \times Y$$

from f to g , $g(q) = G(q, 1)$. We note the properties

$$(iv) \quad g(Q_\alpha^0) \subset \Delta_\alpha \times W_\alpha \subset U, \alpha \in A.$$

$$(v) \quad G(p, t) = f(p), p \in P.$$

$$(vi) \quad \pi_X \circ G(q, t) = f(q), q \in Q.$$

Step 3. For each n -simplex Δ_α , let $u_\alpha : \Delta_\alpha \times I \rightarrow \Delta_\alpha$ be a homotopy such that.

$$u_\alpha(x, 0) = x, u_\alpha(x, t) = x \text{ if } x \in \partial\Delta_\alpha, u_\alpha(x, 1) \in \partial\Delta_\alpha \text{ if } x \in \Delta_\alpha \setminus \Delta_\alpha^0$$

We use u_α to define a homotopy $K : Q \times I \rightarrow X \times Y$ of the map $g : Q \rightarrow X \times Y$ of step 2:

$$\begin{aligned} K(q, t) &= g(q) \text{ if } q \in Q \setminus \bigcup_{\alpha \in A} \text{int}\Delta_\alpha \times Y \\ \pi_X K(q, t) &= u_\alpha(\pi_X \circ g(q), t), q \in g^{-1}(\Delta_\alpha \times Y) \\ \pi_Y K(q, t) &= \pi_Y g(q), q \in g^{-1}(\Delta_\alpha \times Y) \end{aligned}$$

The new map $k(q) = K(q, 1)$ maps $\bigcup_\alpha g^{-1}(\Delta_\alpha^0 \times Y)$ into U and $\pi_X \circ k$ maps $Q \setminus \bigcup_\alpha \Delta_\alpha^0$ into X^{n-1} .

Set $Q_1 = Q \setminus k^{-1}(\bigcup_{\alpha \in A} \text{int}\Delta_\alpha^0 \times Y)$, and $P_1 = P \cap Q_1 \cup \bigcup_\alpha k^{-1}(\partial\Delta_\alpha^0)$.

Then we have the diagram

$$\begin{array}{ccc} P_1 & \subset & Q_1 \\ \downarrow & & \downarrow k \\ U \cap (X^{n-1} \times Y) & \subset & X^{n-1} \times Y. \end{array}$$

By the inductive assumption we can find a homotopy of k to a map $Q_1 \rightarrow U \cap (X^{n-1} \times Y)$ such that the homotopy is constant on $P_1 \times I$ and hence extends to all of Q . This completes the proof.

4 Metrizable of $\Psi_d(\mathbb{R}^{d+n})$

This section proves theorem 2.1, which states that $\Psi_d(\mathbb{R}^{d+n})$ is metrizable in the topology defined in § 1. In 4.1 we give an equivalent definition of the topology and show that $Psi_d(\mathbb{R}^{d+n})$ is a regular space. In the following § ?? we show that the topology is also countable, and hence by a standard theorem that Psi_d^{d+n} is metrizable.

4.1 Redefinition of the topology

Recall that

$$\Psi_d(\mathbb{R}^{d+n}) = \{W^d \subset \mathbb{R}^{d+n} \mid \partial W = \emptyset, W \text{ a closed subset}\}.$$

Suppose that $W \in \Psi_d(\mathbb{R}^{d+n})$. To a point $x \in W$ we associate the point in $G(d, n)$, the Grassmannian of d -planes in \mathbb{R}^{d+n} . The point is determined by the tangent space of W at x . This gives a map

$$W \rightarrow \mathbb{R}^{d+n} \times G(d, n).$$

Chose a metric μ on the compact manifold $G(d, n)$, and consider the product metric μ_2 on $\mathbb{R}^{d+n} \times G(d, n)$.

For every positive $r \in \mathbb{R}$, let rD^{d+n} be the closed disc around the origin of radius r . Let $V, W \in \Psi_d(\mathbb{R}^{d+n})$. We want to say that V and W are close if there is a diffeomorphism ϕ between $V \cap rD^{d+n}$ and $W \cap rD^{d+n}$, such that p is close to $\phi(p)$ and $T_p(W)$ is close to $T_{\phi(p)}(V)$ in the chosen metric on the Grassmannian space. But since a diffeomorphism doesn't necessarily preserve the distance to the origin, we are going to use a more careful formulation.

Instead, we consider diffeomorphisms $\phi : U \rightarrow s(U)$ where $U \subset V, \phi(U) \subset W$ are open subsets. Let $\mathcal{P}_r(V, W)$ be the set of such diffeomorphisms ϕ that satisfy $V \cap rD^{d+n} \subset U$ and $W \cap rD^{d+n} \subset \phi(U)$. When we measure the distance between p and $\phi(p)$, we will only care about points $p \in s(\phi, r) = (V \cap rD^{d+n}) \cup (\phi^{-1}(W \cap rD^{d+n}))$. So, for a diffeomorphism $\phi \in \mathcal{P}_r(M, N)$ we define $d'_r(\phi) = \sup_{p \in s(\phi, r)} \mu_2(p, \phi(p))$. If $s(\phi) = \emptyset$, this number is understood to be 0. Finally, we define

$$d_r(V, W) = \inf_{\phi \in \mathcal{P}_r} (d'_r(\phi)) \in \mathbb{R}_+ \cup \{\infty\}. \quad (11)$$

Again, if $\mathcal{P}_r(V, W) = \emptyset$, this number is understood to be ∞ .

Remark 4.1. The notation above is slightly abusive, since the definition of $d'_r(\phi)$ actually involves the source U of ϕ . This dependence is only weak. In the definition of d' we only use the values of ϕ in the closed subset $s(\phi, r)$. This means that if V is an arbitrary small neighbourhood of $s(\phi, r)$ in U , then $d'_r(\phi|_V) = d'_r(\phi)$.

For example, suppose that $d_r(V, W) < d$. We can find $\phi : U \rightarrow \phi(U)$ such that $d'_r(\phi) < d$. As above, by restricting ϕ to a neighbourhood of $s(\phi, r)$ in U , we can assume that U satisfies that $\mu_2(p, \phi(p)) < d$ for all $p \in U$.

Here are the main properties of the functions d_r :

Lemma 4.2. *Let $W_i \in \Psi_d(\mathbb{R}^{d+n})$. Then*

(i) *Symmetry:* $d_r(W_1, W_2) = d_r(W_2, W_1)$.

(ii) *Semi-continuity:* *If $r \leq r'$, then $d_r(W_1, W_2) \leq d_{r'}(W_1, W_2)$. If $d_r(W_1, W_2) < \epsilon$, there are $\delta, \epsilon_1 > 0$ so that $d_{r+\delta}(W_1, W_2) < \epsilon - \epsilon_1$.*

(iii) *Weak triangle inequality: Given numbers $r_{23} > r_{12} > r_{13} > 0$, the triangle inequality*

$$d_{r_{13}}(W_1, W_3) \leq d_{r_{12}}(W_1, W_2) + d_{r_{23}}(W_2, W_3)$$

is valid for triples of manifolds $\{W_1, W_2, W_3\}$ that satisfy the additional assumption that $d_{r_{12}}(W_1, W_2) < r_{23} - r_{12}$ and $d_{r_{23}}(W_2, W_3) < r_{12} - r_{13}$.

The semi-continuity property says that if we fix W_1 and W_2 , the function $f(r) = d_r(W_1, W_2)$ is a monotonously increasing upper semi-continuous function of r . It is not continuous in general.

The functions d_r do not satisfy the usual triangle inequality. But they do satisfy the weak form of the triangle inequality above, involving several r . There is the technical difficulty that this triangle inequality is only true if the manifolds are close to each other, that is if they satisfy the additional assumption of 4.2.(iii). Fortunately it turns out that these restrictions are not so important when you study the topology defined by the all the functions d_r .

Prof of lemma 4.2. (i) The definition of d_r is symmetric in W_1 and W_2 , exchanging ϕ for ϕ^{-1} .

(ii) Choose $\epsilon_1 > 0$ so that $d_r(W_1, W_2) < \epsilon - \epsilon_1$. Let $\phi : U \rightarrow \phi(U)$ be in $\mathcal{P}_r(W_1, W_2)$ such that for $p \in (W_1 \cap rD^{n+s}) \cup \phi^{-1}(W_2 \cap rD^{n+s})$ we have that $\mu_2(p, \phi(p)) < \epsilon - \epsilon_1$.

Since $U \subset W_1$ is open, it follows that the set $\{p \in U | \mu_2(p, \phi(p)) < \epsilon - \epsilon_1\}$ is open in W_1 . That is, for some $\delta > 0$ it contains $(W_1 \cap (r + \delta)D^{n+s}) \cup \phi^{-1}(W_2 \cap (r + \delta)D^{n+s})$. But then $\phi \in \mathcal{P}_{r+\delta}(W_1, W_2)$, and the statement follows.

(iii) Consider three manifolds W_1, W_2, W_3 which satisfy the conditions $r_{23} - r_{12} > d_{r_{12}}(W_1, W_2)$ and $r_{12} - r_{13} > d_{r_{23}}(W_2, W_3)$. Let d_{12} and d_{23} be real numbers such that $r_{23} - r_{12} > d_{12} > d_{r_{12}}(W_1, W_2)$ and $r_{12} - r_{13} > d_{23} > d_{r_{23}}(W_2, W_3)$.

In order to prove (3), we will show that $d_{12} + d_{23} \geq d_{r_{13}}(W_1, W_3)$. Using remark 4.1 we can find $\phi_{12} : U_1 \rightarrow \phi_{12}(U_1)$ in $\mathcal{P}_{r_{12}}(W_1, W_2)$ and $\phi_{23} : U_2 \rightarrow \phi_{23}(U_2)$ in $\mathcal{P}_{r_{23}}(W_2, W_3)$ such that for all $p \in U_3$ respectively for all $q \in U_2$ we have that $\mu_2(p, \phi_{23}(p)) < d_{23}$ and $\mu_2(q, \phi_{12}(q)) < d_{12}$.

In order to bound $d_{r_{13}}(W_1, W_3)$ from above, we need to construct an element in $\mathcal{P}_{r_{13}}(W_1, W_3)$. The obvious choice to try is the composition $\phi_{23} \circ \phi_{12}$. One problem is that this composition might not be defined on all of U_1 . The first step of the argument is to check that the composition is defined at least on an open neighbourhood of $s(\phi_{12}, r_{12})$ in U_1 .

What makes this work is that for $p \in U_1$ we are assuming that $\mu_2(p, \phi_{12}(p)) < r_{23} - r_{12}$. Because it follows from this and the triangle inequality for μ_2 that if $p \in U_1 \cap r_{12}D^{n+s}$, then $\phi_{23}(p) \in r_{23}D^{n+s}$.

In particular, since $W_2 \cap r_{23}D^{n+s} \subset U_2$ we conclude that possibly after replacing U_1 by a smaller open subset containing $s(\phi_{23}, r_1)$, we can assume that $\phi_{12} \circ \phi_{23}$ is defined on U_1 .

We now claim that $\phi_{23} \circ \phi_{12} \in \mathcal{P}_{r_{13}}(W_1, W_3)$. What we have to prove is that $W_3 \cap r_{13}D^{s+n} \subset \phi_{23} \circ \phi_{12}(U_1)$. But if $q \in W_3 \cap r_{13}D^{s+n} \subset W_3 \cap r_{12}D^{s+n}$ then $q = \phi_{23}(p)$ for

some $p \in U_2$. Since $\mu_2(p, \phi_{23}(p)) < r_{12} - r_{13}$, we have that $p \in W_2 \cap r_{12}D^{s+n} \subset \phi_{12}(U_1)$, so $q = \phi_{23}(p) \in \phi_{23} \circ \phi_{12}(U_1)$.

Finally, the triangle inequality for μ_2 shows that if $p \in s(\phi_{23} \circ \phi_{12}, r_3)$, then

$$\mu_2(p, \phi_{23} \circ \phi_{12}(p)) \leq \mu_2(p, \phi_{12}(p)) + \mu_2(\phi_{12}(p), \phi_{23} \circ \phi_{12}(p)) \leq d_{12} + d_{23}.$$

This proves ((iii)) and completes the proof. \square

Given $M \in \Psi_d(\mathbb{R}_d^{d+n})$ we define the neighborhoods

$$\mathcal{U}_{r,\epsilon}(M) = \{N \in \Psi^d(\mathbb{R}^{d+n}) \mid d_r(M, N) < \epsilon\}.$$

Lemma 4.3. *The sets $\mathcal{U}_{r,\epsilon}(M)$ form a basis for a topology on $\Psi^d(\mathbb{R}^{d+n})$. This topology is regular.*

Proof. To show that the sets $\mathcal{U}_{r,\epsilon}(M)$ form the basis of a topology, we need to show that if $N \in \mathcal{U}_{r_1,\epsilon_1}(M_1) \cap \mathcal{U}_{r_2,\epsilon_2}(M_2)$, then there exist $\epsilon', r > 0$ such that

$$\mathcal{U}_{r',\epsilon'}(M) \subset \mathcal{U}_{r_1,\epsilon_1}(M_1) \cap \mathcal{U}_{r_2,\epsilon_2}(M_2).$$

It is enough to show that if $N \in \mathcal{U}_{r,\epsilon}(M)$, then there exists r' and ϵ' such that $\mathcal{U}_{r',\epsilon'}(N) \subset \mathcal{U}_{r,\epsilon}(M)$.

By lemma 4.2.(ii) there are $\delta, \epsilon_1 > 0$ so that $N \in \mathcal{U}_{r+\delta,\epsilon-\epsilon_1}(M)$. Choose a positive $\epsilon' < \min(\delta, \epsilon_1)$, and put $r' = r + \delta + \epsilon - \epsilon_1$. We claim that $\mathcal{U}_{r',\epsilon'}(N) \subset \mathcal{U}_{r,\epsilon}(M)$. Let $N' \in \mathcal{U}_{r',\epsilon'}(N)$. Use lemma 4.2.(iii) with $W_1 = M$, $W_2 = N$, $W_3 = N'$, $r_{12} = r + \delta$, $r_{23} = r'$ and $r_{13} = r$. The conclusion is that $d_r(M, N') < \epsilon$, which proves our claim.

To prove regularity, it suffices to show that for every $r, \epsilon > 0$ we separate M from the complement of $\mathcal{U}_{r,\epsilon}(M)$. To do this, it suffices to find an open neighbourhood U of M , and for every N in the complement of $\mathcal{U}_{r,\epsilon}(M)$ an open neighbourhood V_N of N disjoint from U .

We chose $U = \mathcal{U}_{r+1,\epsilon/2}(M)$. Let $r' = r + 1 + \epsilon$ and $\epsilon' = \min(\epsilon/2, 1)$. For $N \notin \mathcal{U}_{r,\epsilon}(M)$, we chose $V_N = \mathcal{U}_{r',\epsilon'}(N)$. We need to show that $U \cap V_N = \emptyset$. So assume to the contrary that $N' \in U \cap V_N$. From lemma 4.2.(iii) with $W_1 = M$, $W_2 = N'$, $W_3 = N$, $r_{12} = r + 1$, $r_{23} = r + 1 + \epsilon/2$ and $r_{13} = r$ we obtain that $d_r(M, N) < \epsilon$, in contradiction to the assumption on N . \square

4.2 Countability of the topology

Theorem 4.4. *The topology of $\Psi_d(\mathbb{R}^{n+s})$ has a countable basis.*

We remind the reader that the elements $M \in \Psi_d(\mathbb{R}^{n+s})$ are smooth d -dimensional submanifolds which are closed sets in \mathbb{R}^{d+n} . We will need to introduce a list of curvature conditions of $M \subset \mathbb{R}^{d+n}$. Loosely speaking, these conditions will bound a measure of curvature from above by K , but only on a normal tube around M of radius δ and inside a disc rD^{d+n} . The set of manifolds satisfying the conditions with respect to the numbers $r, \delta, K > 0$ form a set $\mathcal{X}_{r,\delta}(K) \subset \Psi_d(\mathbb{R}^{d+n})$. We defer the precise formulation

of the curvature conditions to later. Instead we formulate the properties of the sets $\mathcal{X}_{r,\delta}(K)$ as lemma 4.5. Then we reduce the proof of theorem 4.4 to lemma 4.5, and finally we discuss the proof of the lemma.

For M, N in Ψ_d^{d+n} , we define the r -Hausdorff distance as

$$d_r^H(M, N) = \max\left(\sup_{x \in M \cap rD^{n+s}} d(x, N), \sup_{x \in N \cap rD^{n+s}} (d(x, M))\right).$$

If $d_r^H(M, N)$ is small, the two manifolds are pointwise close to each other after intersecting with rD^{n+s} , but we don't assume that the tangent spaces at close points are close. In general, a bound on the Hausdorff distance between M and N does not give a bound on the distance $d_r(M, N)$ defined by (11) in §4.2.

Lemma 4.5. *For $\delta, r, K > 0$ there is a subset $\mathcal{X}_{r,\delta}(K) \subset \Psi_d^{d+n}$ with the following properties.*

- (i) *Given $M \in \Psi_d^{d+n}$ and any $r > 0$ there are $\delta > 0, K > 0$ such that $M \in \mathcal{X}_{r,\delta}(K)$.*
- (ii) *If $r' \leq r, \delta' \leq \delta, K' \leq K$, then $\mathcal{X}_{r',\delta'}(K') \subset \mathcal{X}_{r,\delta}(K)$.*
- (iii) *For any $r, \epsilon, K > 0$ there exists $\delta > 0$ with the following property: If $\delta' < \delta$, $M, N \in \mathcal{X}_{r+1,\delta'}(K)$ and $d_{r+1}^H(M, N) < \delta'$, then $d_r(M, N) < \epsilon$.*

We want to construct a countable, dense set in $\Psi_d(\mathbb{R}^{d+n})$. This set will depend on choices to be specified below.

For the moment, we fix numbers $r, \delta > 0$. Chose a finite set of points $\{x_i\}_{i \in I} \subset rD^{d+n}$, $i \in I$ such that for any $x \in rD^{d+n}$ there is an i such that $d(x, x_i) < \delta/2$. For any $N \in \Psi_d^{d+n}$ we define $S(N) = \{i \in I \mid d(x_i, N) < \delta/2\} \subset I$. If $S(N_1) = S(N_2)$, then obviously $d_r^H(N_1, N_2) < \delta$.

There will be a finite family J of sets $j \subset I$ such that there exists an $N \in \mathcal{X}_{r,\delta}(K)$ with $j = S(N)$. For each $j \in J$ chose a manifold N_j such that $j = S(N_j)$.

Definition 4.6. Let $\mathcal{N}_{r,\delta}(K)$ be this set of manifolds.

Remark 4.7. For any $M \in \mathcal{X}_{r,\delta}(K)$, there is an $N \in \mathcal{N}_{r,\delta}(K)$ such that the Hausdorff distance satisfies $d_r^H(M, N) < \delta$. Actually, we can chose N as the unique $N_j \in \mathcal{N}_{r,\delta}(K)$ such that $S(N_j) = S(M)$.

Lemma 4.8. *The subset*

$$\mathcal{N} = \bigcup_{r,\delta,K \in \mathbb{Q}_+} \mathcal{N}_{r,\delta}(K) \subset \Psi_d(\mathbb{R}^{d+n})$$

is countable and dense.

Proof. \mathcal{N} is countable since it is a countable union of finite sets. We have to prove that it is dense, that is, for any $M \in \Psi^r(\mathbb{R}^{d+n})$ and any $r, \epsilon > 0$ there is an $N \in \mathcal{N}$ such that $d_r(M, N) < \epsilon$.

By 4.5.(i) we have that $M \in \mathcal{X}_{r+1, \delta_0}(K)$ for some rational $r > 0$ and $\delta_0, K > 0$. By 4.5.(ii) we can assume that K is also a rational number. Using 4.5.(iii) we find $\delta_1 > 0$ such that if $M, N \in \mathcal{X}_{r, \delta}(K)$ and $0 < \delta < \delta_1$, then $d_r(M, N) < \epsilon$.

Pick $\delta > 0$ to be a rational number, $\delta < \min(\delta_0, \delta_1)$. Then $M \in \mathcal{X}_{r+1, \delta}(K)$ by 4.5.(ii). According to remark 4.7 there is an $N \in \mathcal{N}_{r, \delta}(K)$ such that $d_r^H(M, N) < \delta$. Since $\delta < \delta_1$, it follows that $d_r(M, N) < \epsilon$, completing the proof of the lemma. \square

Proof of theorem 4.4. We claim that the open sets $\mathcal{U}_{r, \epsilon}(N)$ with $N \in \mathcal{N}$ and rational r, ϵ form a countable basis for the topology. To prove that this is a basis, it suffices to show that for any M, M', r_1, a_1 such that $M \in \mathcal{U}_{r, d}(M')$, there is an $N \in \mathcal{N}$ and $r, \epsilon > 0$ so that $M \in \mathcal{U}_{r, \epsilon}(N) \subset \mathcal{U}_{r_1, a_1}(M')$. By lemma 4.2.(ii), we can find $r_2 > r_1$ and a positive $a_2 < a_1$ such that $d_{r_2}(M, M') < a_2$.

Using lemma 4.2.(iii) we can find $a, b > 0$ such that if $N \in \mathcal{U}_{r_2+b, a}(M)$, then $M \in \mathcal{U}_{r_2+b, a}(N) \subset \mathcal{U}_{r_1, a_1}(M')$. For instance, $a = (a_1 - a_2)/2$ and $b = \max((r_2 - r_1)/2, a_1)$ will do. Since \mathcal{N} is dense, we can find such an $N \in \mathcal{N}$, which concludes the proof of the theorem. \square

We now need to define the class $X_{r, \delta}(K)$. To define this class, we write down a sequence of conditions on a manifold M . These conditions depend on the positive numbers r, δ and K . For any manifold embedded in \mathbb{R}^{n+s} , we consider the exponential map

$$E_M : \nu_\delta(M) \rightarrow \mathbb{R}^{n+s} \quad E_M(p, v) = v + p.$$

Condition 1. If we restrict E_M to $\{(p, v) \in \nu M; |p| < r, |v| < \delta\}$ this map becomes a diffeomorphism onto its image.

Condition 2. The normal curvature of M is bounded by K in $M \cap rD^{n+s}$.

The inverse of E_M followed by projection to M is a differentiable map

$$F : E_M(\nu_\delta) \rightarrow M.$$

The last condition is concerned with this map. We consider F as a map from an open subset of \mathbb{R}^{n+s} to \mathbb{R}^{n+s} . In particular, we can define arbitrary partial derivatives of F . For any given manifold M , these derivatives are bounded on any compact subset.

Condition 3. All first and second order derivatives of F are bounded by K , that is

$$\left| \frac{\partial F_i}{\partial x_j}(x) \right| < K, \quad \left| \frac{\partial^2 F_i}{\partial x_j \partial x_k}(x) \right| < K \quad \text{for } x \in E_M(\nu_\delta|_{M \cap rD^{n+s}})$$

The final condition is topological.

Condition 4. If $\epsilon < 4\delta$ and $p \in M$, then $M \cap \text{int} D_\epsilon(p)$ is contractible.

Definition 4.9. $X_{R, \delta}(K)$ is the set of manifolds $M \in \psi_d(\mathbb{R}^{d+n})$ satisfying the above conditions 1–4.

We now turn to

Proof of lemma 4.5. The first two statements of the lemma are easy to verify, but we do have to prove the third statement. So let $r, \epsilon, K > 0$ be given. We need to specify a $\delta > 0$.

Suppose that $M \in X_{r+1, \delta}(K)$ and that N has normal curvature less than K . Further suppose that $N \cap (r+1)D^n$ is contained in an δ tubular neighbourhood of M . This defines a projection map $\pi : N \rightarrow M$. If $\delta < \epsilon/2$ then $d(x, \pi(x)) < \epsilon/2$.

We want to show that we can choose δ so that the distance in the Grassmannian manifold between $T_p(N)$ and $T_{\pi(x)}(M)$ arbitrary smaller than $\epsilon/2$ for $p \in N \cap rD^{n+s}$.

Let $p \in N \cap rD^{n+s}$ and $v \in T_p(N)$ a unit length tangent vector. Pick a geodesic γ in N so that $\gamma(0) = p$ and $\gamma'(0) = v$. The geodesic is defined for all t , and if $|t| < 1$ we have that $\gamma(t)$ is contained in an δ tubular neighbourhood of M . Let $f(t) = \gamma(t) - F\gamma(t)$. By elementary calculation, the second derivative of the components of f satisfies

$$f''(t)_i = \gamma''_i - \sum_{j,k} \frac{\partial^2 F_i}{\partial x_j \partial x_k} \gamma'_j \gamma'_k - \sum_j \frac{\partial F_i}{\partial x_j} \gamma''_j$$

Recall that γ is parametrized by arc length and has curvature bounded by K . By condition 3, $|f''(t)_i| \leq K_1$ where $K_1 = K + K^2 + K^3$.

So we obtain:

$$\begin{aligned} |f(t)_i| &< \delta \\ |f''(t)_i| &< K_1. \end{aligned} \tag{12}$$

By elementary arguments, if a twice differentiable function defined on $[-1, 1]$ satisfies (12), it follows that $|f'(0)| < \max(2\delta, 2\sqrt{\delta K_1})$. This is a special case of the Landau-Kolmogorov inequalities. Given K and thus K_1 , by choosing δ small enough, we can assure that $|f'(0)| = |v - (F\gamma)'(0)|$ can be made arbitrarily small.

But since $(F\gamma)'(0)$ is contained in the tangent space of M at $F(p)$, it follows that the projection of v to the normal space of M at $F(p)$ can also be made arbitrarily small.

Since v is a unit vector, we see that we can chose δ so small that the distance between $T_p N$ and $T_{F(p)}(M)$ can also be made smaller than $\epsilon/2$ in the Grassmannian space.

It follows that the map $F_M : N \rightarrow M$ satisfies that $d_r(x, F(x)) < \epsilon$. To finish the proof, we have to show that F is a diffeomorphism on its image.

At least we know that the map $F_M : N \rightarrow M$ is a local diffeomorphism, since its differential is the projection of $T_p M$ to $T_{F(p)} N$. Let $U = N \cap (r+1-\delta)\text{int} D^{n+s}$. We have to show that F_M is injective on U , and that its image is an open set in M which contains $M \cap rD^{n+s}$.

It's easy to see that $F_M(U) \cap (r+\delta)D^{n+s}$ is open and closed in $M \cap (r+\delta)D^{n+s}$, so the image consists of a union of components. If $p \in M \cap rD^{n+s}$, there is a point $q \in N$ such that $d(p, q) < \delta$ and $d(F_M(q), p) < 2\delta$. By condition 4, p and $F_M(q)$ are in the same component of $M \cap (r+\delta)D^{n+s}$. It follows that $p \in F_M(U)$.

Finally we need to prove that F_M is injective on U . Suppose $p, q \in U$, and $F_M(p) = F_M(q)$. There is a curve connecting p and q of diameter less than 2δ . Its image in M is

a closed curve of diameter less than 4δ , so it is null-homotopic. Since F_M is a covering map, $p = q$. \square

Lemma 4.10. *The topology we have defined on $\Psi^d(\mathbb{R}^{d+n})$ has a second countable basis. Moreover, $\Psi^d(\mathbb{R}^{d+n})$ is a separable, metrizable topological space.*

Proof. We prove the following statement: For every $\epsilon, r, C > 0$ there is a countable subset $X \subset \Psi(\mathbb{R}^{d+n})$ such that for every $W \in \Psi(\mathbb{R}^{d+n})$ whose normal curvature is bounded by C there is an $M \in X$ such that $d_r(W, M) < \epsilon$.

Given this, we can find a countable set X such that for every $r, \epsilon > 0$ and W we can find an $M \in X$ such that $d_r(M, W) < \epsilon$. This set is clearly dense, and using lemma 4.2 we see that the sets $\{\mathcal{U}_{r,\epsilon}(M) \mid M \in X; r, \epsilon \in \mathbb{Q}\}$ form a countable basis for the topology. It follows from Urysohn's metrisation theorem that the topology is metrizable. By lemma 4.8 it is separable. \square

4.3 Equivalence of topologies.

We need to compare the topology of lemma 4.3 to the topology defined in section 1.

For $M \subset \mathbb{R}^{d+n}$ the fine \mathcal{C}^1 topology on the space of \mathcal{C}^1 maps $f : N \rightarrow M$ as the topology generated by the sets

$$f : N \rightarrow M; \quad |f(x) - g(x)| < \delta(x), |df_x(v) - dg_x(v)| < \delta(x)|v|$$

where $\delta : M \rightarrow \mathbb{R}$ is a positive, continuous function. See [18], definition 3.5.

We will need:

Theorem 4.11 ([18], theorem 3.10). *Let $M \rightarrow N$ be a \mathcal{C}^1 map. If f is a diffeomorphism, there is a fine neighbourhood of f such that if g is in this neighbourhood, then g is a diffeomorphism.*

We will need the following elementary estimate.

Lemma 4.12. *Let $V \subset \mathbb{R}^{d+n}$. Let $A : V \rightarrow V, A^\perp : V \rightarrow V^\perp$ be linear maps. Let $W = (\text{Id} + A, A^\perp)V \subset \mathbb{R}^{d+n}$. For any $\epsilon > 0$ there are numbers $\delta_1 > 0, \delta_2 > 0$ such that if $|A| < \delta_1$ and in the metric of the Grassmannian $d(V, W) < \delta_2$, then $|(A, A^\perp)| < \epsilon$. \square*

Proof of lemma 2.1. We need to show that the topology \mathcal{T} defined by the sets $\mathcal{N}_{K,\epsilon'}(W)$ agrees with the topology on $\Psi^d(\mathbb{R}^{d+n})$ we have defined above.

We first show that \mathcal{T} is finer than the topology of $\Psi^d(\mathbb{R}^{d+n})$.

For any W, r, ϵ we can find K, ϵ' so that $W \in \mathcal{N}_{K,\epsilon'}(W) \subset \mathcal{U}_{r,\epsilon}(W)$. If $s : M \rightarrow \mathbb{R}^{d+n}$ is defined by a small section of the normal bundle we put (as above) $\phi(x) = x + s(x)$. Then ϕ is a diffeomorphism from M to W . A vector $v \in T_x M \subset \mathbb{R}^{d+n}$ is close to the corresponding vector $v + ds_x(v) \in T_{\phi(x)}(s(M)) \subset \mathbb{R}^{d+n}$. It follows that the vectorspace $T_{s(x)}(s(M))$ is close to $T_x(M)$ in the Grassmannian, and that we can choose ϵ so small that $\mathcal{N}_{K,\epsilon'}(W) \subset \mathcal{U}_{r,\epsilon}(W)$.

In the general case, if $M \in \mathcal{U}_{r,\epsilon}(W)$, we can use lemma 4.2.(ii) to find r', ϵ' so that $M \in \mathcal{U}_{r',\epsilon'}(M) \subset \mathcal{U}_{r,\epsilon}(W)$, and apply the above argument to $M \in \mathcal{U}_{r',\epsilon'}(M)$. It follows that the topology defined by the $\mathcal{N}_{K,\epsilon'}(W)$ is at least as fine as the topology on $\Psi^d(\mathbb{R}^{d+n})$.

We have to prove the opposite implication. Given W, K, ϵ' we need to find r, ϵ so that

$$W \in \mathcal{U}_{r,\epsilon}(W) \subset \mathcal{N}_{K,\epsilon'}(W).$$

If ϵ is sufficiently small (depending on W and a number $r > 0$), the map

$$e : \nu(W \cap r \text{int} D^{d+n}) \rightarrow \mathbb{R}^{d+n}, \quad (x, v) \mapsto x + v$$

is a diffeomorphism onto its image.

It's inverse followed by the projection defines a differentiable map π , where $\pi(x)$ denotes the unique point on W which is closest to x . If ϵ is sufficiently small, the composite $\pi \circ s$ will be close to the identity in the \mathcal{C}^1 topology.

It follows from theorem 4.11 that $\pi \circ s$ is a diffeomorphism onto its image. The inverse of π is given by a section s in the normal bundle of M , and $W = \pi^{-1}(M)$. We still need to show that possibly after decreasing ϵ we can make the norm of s arbitrarily small.

For each $x \in M$, we can write the differential ds as a sum $ds_\tau \oplus ds_\nu$ where $ds_{x\tau} \in T_x(M)$ and $ds_{x\nu} \in \nu_x(M)$.

Suppose that the tangent bundle of M has a family of sections t_i , forming an orthogonal basis at each point. Since $\langle s, t_i \rangle = 0$ we have that for any section s ,

$$|ds_{x\tau}(v)|^2 = \left(\sum_i \langle ds(v), t_i \rangle \right)^2 \leq |s|^2 \left(\sum_i |dt_i(v)| \right)^2$$

Let C be a constant such that $C^2 \geq (\sum_i |dt_i|)^2$. Then $|ds_\tau| \leq C\epsilon$. Using lemma 4.12, we see that by making ϵ sufficiently small, we can ensure that the norm of the section s is arbitrarily small. Finally, cover $M \cap r D^{d+n}$ by a finite number of open sets, so that the tangent bundle of M has a family of orthonormal sections on each of these open sets, and repeat the argument for each of these open sets. \square

5 Relation to A-theory

This section describes a relation between the classifying spaces of the embedded cobordism categories and Waldhausen's A -theory. More precisely we shall describe a map

$$\tau : \Omega B\mathcal{C}_{d,n} \rightarrow A(G(d, n))$$

where $A(G(d, n))$ is Waldhausen's K -theory of the Grassmannian $G(d, n)$. The map is an infinite loop map if $n = \infty$.

5.1 A convenient model for A -theory

Recall first the standard definition of $A(X)$ from [23]. Let $R_{hf}(X)$ be the category of homotopy finite retractive spaces over X . We work in the category of compactly generated Hausdorff spaces. It has objects (Y, r, s) where $r : Y \rightarrow X$, $s : X \rightarrow Y$ and $rs = \text{id}_X$, such that $(Y, s(X))$ is homotopy equivalent to a finite CW complex relative to $s(X)$. This is a category with cofibrations and weak equivalences, that is a Waldhausen category.

A map $i : Y_1 \rightarrow Y_2$ over X is a cofibration if the underlying map (forgetting X) is a cofibration. Since we work in the category of Hausdorff spaces all cofibrations are closed cofibrations([2]). It is a weak equivalence if $i_* : \pi_k(Y_1) \rightarrow \pi_k(Y_2)$ is a bijection for all k .

Let $S_\bullet(X)$ denote Waldhausen's S_\bullet construction applied to $R_{hf}(X)$. An element of $S_q(X)$ is a flag

$$X \xrightarrow{s} Y_1 \rightharpoonup Y_2 \cdots \rightharpoonup Y_q \xrightarrow{r} X,$$

together with a choice of quotients $Y_j/Y_i := Y_j \cup_{Y_i} X$, such that $X \rightarrow Y_i \rightarrow X$ is in $R_{hf}(X)$. The graded set $S_\bullet(X)$ is a simplicial set (d_0 divides out Y_1 while d_j omits Y_j when $j \geq 1$). Weak equivalences of flags define a simplicial category $wS_\bullet(X)$ whose nerve is the bi simplicial set $N_\bullet^w S_\bullet(X)$, and

$$A(X) = \Omega |N_\bullet^w S_\bullet(X)|. \quad (13)$$

In order to compare the embedded cobordism category with A -theory we need a variant of Waldhausen's construction which we now turn to. Let B be a locally compact CW-complex satisfying (B2) below. Let

$$W(X, B) \subset R_{hf}(X) \quad (14)$$

be the subset of retractive space (Y, r, s) over $X \times B$ with the two extra requirements:

(B1) $Y \xrightarrow{r} X \times B \xrightarrow{\text{id}_B} B$ is a fibration,

(B2) $Y \xrightarrow{\Delta} Y \times Y$ is a cofibration.

We use the term fibration to mean a surjective Horowitz fiber space. A Hausdorff space which satisfies (B2) is called locally unconnected(LEC), see [4], [13] for a discussion of this category. We also refer the reader to [16], in particular chapter 4.

Lemma 5.1. *For X and B LEC, $W(X, B)$ is a Waldhausen subcategory of $R_{hf}(X \times B)$, provided B is a locally compact CW complex.*

Proof. We must verify the axioms of [23]. Only the cofibration axioms needs to be checked. The initial object is $X \times B$ which is LEC, since both X and B are. For $(Y, r, s) \in W(X, B)$ the map $X \times B \xrightarrow{s} Y$ is the inclusion of a retract of an LEC and hence a cofibration [4](theorem II.7), [13](lemma 2.17). For cobase change: Given

$$(Y_2, r_2, s_2) \xleftarrow{f} (Y_0, r_0, s_0) \xrightarrow{i} (Y_1, r_1, s_1)$$

in $W(X, B)$ we must check that the adjunction space $(Y_2 \cup_f Y_1, r_2 \cup_f r_1, s_2 \cup_f s_1)$ is in $W(X, B)$. The total space $Y_2 \cup_f Y_1$ is LEC by the adjunction theorems of [4] or [13](theorem 2.3). Finally

$$r_2 \cup_f r_1 : Y_2 \cup_f Y_1 \rightarrow B$$

is a fibration by [1](theorem 2.5). \square

We define $A(X, B)$ to be the K -theory of $W(X, B)$,

$$A(X, B) := \Omega |N_\bullet^w S_\bullet(W(X, B))|. \quad (15)$$

It is equal to $A(X)$ when B is a one-point space. We next examine $A(X, B)$ for fixed X and varying B .

For a map $f : B_1 \rightarrow B_2$ we shall construct a contravariant functor

$$f^* : W(X, B_2) \rightarrow W(X, B_1) \quad (16)$$

and, provided that f is a fibration, also a covariant functor

$$f_* : W(X, B_1) \rightarrow W(X, B_2). \quad (17)$$

The functors are “exact” in the sense that they preserve cofibrations and weak equivalences - they are functors of Waldhausen categories.

The contravariant functor is defined via pullback under $\text{id} \times f : X \times B_1 \rightarrow X \times B_2$. Given $(Y_2, r_2, s_2) \in W(X, B_2)$, let

$$Y_1 = \{(y_2, x, b_1) | (x, f(b_1)) = r_2(y_2)\}$$

This is the total space of the pull-back by f of the fibration $Y_2 \rightarrow B_2$, so Y_1 is LEC, [8]. There are obvious maps $r_1 : Y_1 \rightarrow X \times B_1$ and $s_1 : X \times B_1 \rightarrow Y_1$ defining an element of $W(X, B)$.

Lemma 5.2. *The pull-back $f^* : W(X, B_2) \rightarrow W(X, B_1)$ is a functor of Waldhausen categories.*

Proof. Let $(Y_2, r_2, s_2) \rightarrow (Y'_2, r'_2, s'_2)$ be a cofibration in $W(X, B)$. Since Y_2 and Y'_2 fibers over B , it follows from [9] that $i : Y_2 \rightarrow Y'_2$ is a cofibration over B , that is, there is a fibrewise retraction

$$\begin{array}{ccc} Y'_2 \times I & \xrightarrow{\pi_2} & Y'_2 \times \{0\} \cup Y_2 \times I \\ & \searrow & \swarrow \\ & B_2 & \end{array}$$

of the obvious inclusion. Let $(Y_1, r_1, s_1) = f^*(Y_2, r_2, s_2)$ and $(Y'_1, r'_1, s'_1) = f^*(Y'_2, r'_2, s'_2)$. Then

$$\pi_1(y'_2, t, b_1) := (\pi_2(y'_2, t), b_1)$$

defines a retraction $Y'_1 \times I \rightarrow Y_1 \times \{0\} \cup Y_1 \times I$ Hence f^* preserves cofibrations.

The functor f^* also preserves weak equivalences since it maps fibrations to fibrations. \square

The covariant structure f_* is induced from

$$f_* := (X \times f)_* : R_{hf}(X \times B_1) \rightarrow R_{hf}(X \times B_2)$$

that sends (Y_1, r_1, s_1) to (Y_2, r_2, s_2) with

$$Y_2 = X \times B_2 \cup_{X \times f} Y_1$$

We must show that it defines an element of $W(X, B_2)$. This follows from the references and arguments above. We remark that $f : B_1 \rightarrow B_2$ being a fibration implies that $Y_1 \rightarrow B_1 \rightarrow B_2$ is a fibration so that [1] applies to show that $Y_2 \rightarrow B_2$ is a fibration.

Theorem 5.3. *Homotopic maps $f, g : B_1 \rightarrow B_2$ induce homotopic maps*

$$f^* \simeq g^* : A(X, B_2) \rightarrow A(X, B_1)$$

Proof. We will show that the inclusion $i_0 : B \times \{0\} \rightarrow B \times I$ induces a homotopy equivalence

$$i_0^* : |N_\bullet^w S_\bullet X(X, B \times I)| \rightarrow |N_\bullet^w S_\bullet X(X, B)|$$

This suffices since the projection $B \times I \rightarrow B$ will also induce a homotopy equivalence, and one can compose with the homotopy $B_1 \times I \rightarrow B_2$ to complete the proof.

Let $h : I \rightarrow I$ be the constant map at 0. We must prove that it induces a homotopy equivalence h^* of $|N_\bullet^w S_\bullet X(X, B \times I)|$.

This follows if we can show that for each n , the functor

$$wS_n(W(X, B \times I)) \rightarrow wS_n(W(X, B \times I))$$

induced by h^* , is connected to the identity by a sequence of natural transformation. Here $wS_n(-)$ denotes the category with objects $S_n(-)$ and weak equivalences as morphisms. To this end consider

$$\begin{aligned} \mu : I \times I &\rightarrow I, \mu(s, t) = st \\ \pi : I \times I &\rightarrow I, \pi(s, t) = s, \end{aligned}$$

and the canonical functor

$$H^* : W(X, B \times I) \xrightarrow{\mu^*} W(X, B \times I \times I) \xrightarrow{\pi_*} W(X, B \times I)$$

Let $i_0, i_1 : I \rightarrow I \times I$ be the maps $i_\nu(s) = (s, \nu)$. There is an induced diagram

$$\begin{array}{ccccc} & & W(X, B \times I) & & \\ & \nearrow \text{id} & \uparrow i_1^* & \nwarrow \text{id} & \\ H^* : W(X, B \times I) & \xrightarrow{\mu^*} & W(X, B \times I \times I) & \xrightarrow{\pi_*} & W(X, B \times I) \\ & \searrow h^* & \downarrow i_0^* & \nearrow \text{id} & \\ & & W(X, B \times I) & & \end{array}$$

The three functors from $W(X, B \times I)$ to itself are connected by natural transformations

$$\text{Id} \rightarrow H^* \leftarrow h^*$$

in the category $wW(X, B \times I)$ with morphisms being weak homotopy equivalences, cf. remark 5.4 below. There is an induced diagram with $W(X, B \times I)$ replaced by $wS_n W(X, B \times I)$ and induced natural transformations. Consequently:

$$h^* \simeq \text{id} : N_\bullet^w S_n(W(X, B \times I)) \quad (18)$$

for all n , so that

$$h^* \simeq \text{id} : A(X, B \times I) \rightarrow A(X, B \times I)$$

by standard simplicial techniques. \square

Remark 5.4. Given $f : B \rightarrow C$, $i : C \rightarrow B$ with $f \circ i = \text{id}$. Let $(Y, r, s) \in W(X, B)$ and $Y_0 = i^*(Y, r, s)$. Then the inclusion

$$Y_0 \rightarrow X \times C \cup_{1 \times f} Y$$

is a weak homotopy equivalence if $f : B \rightarrow C$ is a weak homotopy equivalence.

We now let B vary over the standard simplices Δ^p to get a simplicial space

$$[p] \mapsto |N_\bullet^w S_\bullet(X, \Delta^p)|, \quad (19)$$

where the simplicial maps are induced from the standard face and degeneracy maps $\Delta^p \rightarrow \Delta^q$ via the contravariant structure (16).

It follows from Theorem (5.3) that all structure maps in (19) are weak homotopy equivalences. Thus

Corollary 5.5.

$$|N_\bullet^w S_\bullet(W, \Delta^\bullet)| \simeq |N_\bullet^w S_\bullet(X)|.$$

5.2 The map to A -theory

Recall from section 2.3 the two versions $N_\bullet^\delta \mathcal{C}_{d,n}^k$ and $N_\bullet \mathcal{C}_{d,n}^k$ with weakly equivalent geometric realizations. In this section we define a simplicial map

$$\tau : \text{sin}_\bullet N_\bullet^\delta \mathcal{C}_{d,n}^k \rightarrow S_\bullet^{(k)} W(D(d, n), \Delta), \quad (20)$$

where $\text{sin}_\bullet(X)$ denotes the simplicial set which in degree p consists of singular simplices $\Delta^p \rightarrow X$, $S_\bullet^{(k)}$ is the k -fold iterated S_\bullet -construction and $W(G(d, n), \Delta^p)$ the Waldhausen category defined above.

We start with the case $k = 1$, where we write $\mathcal{C}_{d,n}$ instead of $\mathcal{C}_{d,n}^1$. We must define

$$\tau_{p,q} : \text{sin}_p N_{d,n}^\delta \rightarrow S_q W(G(d, n), \Delta^p)$$

compatible with the bisimplicial structure maps. This requires some preparations about the structure of $N_q^\delta \mathcal{C}_{d,n}$ which we now turn to. See also §2.1 of [6].

Let W^d be an abstract (as opposed to embedded) cobordism from M_0 to M_1 , equipped with disjoint collars

$$h_0 : [0, 1] \times M_0 \rightarrow W, \quad h_1 : [0, 1] \times M_1 \rightarrow W$$

and let $\text{Emb}_\epsilon(W, [0, 1] \times \mathbb{R}^{d+n-1})$ denote the space of smooth embeddings

$$e : W \rightarrow [0, 1] \times \mathbb{R}^{d+n-1}$$

such that there are embeddings $e_\nu : M_\nu \rightarrow \mathbb{R}^{d+n-1}$ with

$$e \circ h_0(t_0, x_0) = (t_0, e_0(x_0)), \quad e \circ h_1(t_1, x_1) = (t_1, e_1(x_1))$$

where $e_0 \in [0, \epsilon)$ and $e_1 \in (1 - \epsilon, 1]$. Similarly, let $\text{Diff}_\epsilon(X)$ be the group of diffeomorphisms that restricts to product diffeomorphisms on the ϵ -collars. We let $\text{Emb}(-, -)$ and $\text{Diff}(-, -)$ denote the colimits as $\epsilon \rightarrow 0$. Define

$$\begin{aligned} E_n(W) &:= \text{Emb}(W, [0, 1] \times \mathbb{R}^{d+n-1}) \times_{\text{Diff}(W)} (W) \\ B_n(W) &:= \text{Emb}(W, [0, 1] \times \mathbb{R}^{d+n-1}) / \text{Diff}(W) \end{aligned}$$

The projection $\pi : E_n(W) \rightarrow B_n(W)$ is a smooth fiber bundle of infinite dimensional smooth manifolds in the convenient topology of [10], in fact an embedded bundle in the sense of the diagram

$$\begin{array}{ccc} E_n(W) & \hookrightarrow & B_n(W) \times \mathbb{R}^{d+n} \\ \downarrow \pi & \swarrow & \\ B_n(W) & & \end{array}$$

Moreover, a smooth map $B^m \rightarrow B_n(W)$ from a finite dimensional manifold B^m induces smooth embedded fiber bundle of finite dimensional manifolds

$$\begin{array}{ccc} E^{m+d} & \hookrightarrow & B^m \times \mathbb{R}^{d+n} \\ \downarrow \pi & \swarrow & \\ B^m & & \end{array},$$

and continuous maps into $B_n(W)$ can be approximated by smooth maps, so the set of homotopy classes of continuous maps from B^m to $B_n(W)$ is equal to the set of homotopy classes of continuous maps.

For a closed $(d-1)$ -dimensional manifold M^{d-1} there is a similar smooth fiber bundle of infinite dimensional manifolds

$$\begin{array}{ccc} E_n(M) & \hookrightarrow & B_n(M) \times \mathbb{R}^{d+n-1} \\ \downarrow \pi & \swarrow & \\ B_n(M) & & \end{array}$$

as in §2.1 of [6].

We topologize $N_1^\delta \mathcal{C}_{d,n}$ as the disjoint union of the object space $N_0^\delta \mathcal{C}_{d,n}$ and of the space of non-identity morphisms. Then there is a homeomorphism

$$N_1^\delta \mathcal{C}_{d,n} \cong \coprod_{\{M\}} (B_n(M^{d-1}) \times \mathbb{R}^\delta) \sqcup \coprod_{\{W\}} (B_n(W^d) \times (\mathbb{R}_+^2)^\delta) \quad (21)$$

where the disjoint union is over certain diffeomorphism classes of closed $(d-1)$ -manifolds, respectively compact d -dimensional cobordisms, namely the diffeomorphism classes that embed in \mathbb{R}^{d+n-1} resp. \mathbb{R}^{n+d} . We note that (21) gives $N_1^\delta \mathcal{C}_{d,n}$ the structure of an infinite dimensional smooth manifold.

Let $\sigma : \Delta^p \rightarrow N_1^\delta \mathcal{C}_{d,n}$ be a smooth p -simplex, landing in a non-identity component. This induces a smooth embedded fiber bundle.

$$\begin{array}{ccc} E[a_0, a_1] & \hookrightarrow & \Delta^p \times [a_0, a_1] \times \mathbb{R}^{d+n-1} \\ \downarrow \pi & & \swarrow \\ \Delta^p & \hookleftarrow & \end{array}, \dim E[a_0, a_1] = p + d.$$

For $z \in E[a_0, a_1]$, the vertical tangent space $T_z^\pi E[a_0, a_1]$ is a subspace of $\{\pi(z)\} \times \mathbb{R}^{d+n}$. This defines a map

$$\tau : E[a_0, a_1] \rightarrow G(d, n)$$

into the Grassmannian. Let $E(a_0)$ be the left-hand boundary of $E[a_0, a-1]$ and consider the retractive space

$$G(d, n) \times \Delta^p \xrightarrow{s} E[a_0, a_1] \cup_{E(a_0)} G(d, n) \times \Delta^p \xrightarrow{r} G(d, n) \times \Delta^p \quad (22)$$

with $r = (\tau, \pi)$ on $E[a_0, a_1]$ and the identity on $G(d, n) \times \Delta^p$. The composition

$$E[a_0, a_1] \cup_{E(a_0)} G(d, n) \times \Delta^p \xrightarrow{r} G(d, n) \times \Delta^p \rightarrow \Delta^p$$

is a fibration with LEC total space. Thus (22) defines an element of the Waldhausen category $W(G(d, n), \Delta^p)$. The resulting map

$$\sin_p(N_1^\delta \mathcal{C}_{d,n}) \rightarrow W(G(d, n), \Delta^p)$$

respects the simplicial identities as p varies. Quite similarly, a singular p -simplex of $N_q^\delta \mathcal{C}_{d,n}$ defines a sequence of codimension zero embeddings

$$E[a_0, a_1] \subset E[a_0, a_2] \subset \cdots \subset E[a_0, a_q] \subset \Delta^p \times [a_0, a_q] \times \mathbb{R}^{d+n}$$

fibering over Δ^p with $E[a_0, a_{i+1}] = E[a_0, a_i] \cup_{E(a_i)} E[a_i, a_{i+1}]$. This amounts to a map

$$\tau_{p,q} : \sin_p N_q^\delta \mathcal{C}_{d,n} \rightarrow S_q W(G(d, n), \Delta^p)$$

that gives rise to a bisimplicial map

$$\tau_{\bullet,\bullet} : \sin_{\bullet} N_{\bullet}^{\delta} \mathcal{C}_{d,n} \rightarrow S_{\bullet} W(G(d, n), \Delta^p).$$

We can include $S_{\bullet}(W)$ into $N_{\bullet}^w S_{\bullet}(W)$ and get by corollary 5.5:

$$\tau : |\sin_{\bullet} N_{\bullet}^{\delta} \mathcal{C}_{d,n}| \rightarrow |N_{\bullet}^w S_{\bullet}(G(d, n))|.$$

Finally, the canonical map $|\sin_{\bullet}(X)| \rightarrow X$ is a weak equivalence, and $|N_{\bullet}^{\delta} \mathcal{C}_{d,n}| \sim |N_{\bullet} \mathcal{C}_{d,n}|$ by 2.3. This proves

Theorem 5.6. *Tangents along the fiber induces a weak map*¹

$$\Omega B\mathcal{C}_{d,n} \rightarrow A(G(d, n)).$$

□

The remainder of this section will argue that the map τ in the above theorem is an infinite loop map (when $n = \infty$). Theorems 2.4 and 2.7 imply that the classifying space $BC_{d,n}^k$ is a $(k-1)$ -fold deloop of $B\mathcal{C}_{d,n}$, provided that $k \leq d+n$. For a Waldhausen category \mathcal{C} , the iterated S_{\bullet} -construction $S_{\bullet}^{(k)}$ deloops $S_{\bullet}\mathcal{C}$ by proposition 1.5.3 of [23]. Since $N_{\bullet}^{\delta} \mathcal{C}_{d,n}^k$ is weakly equivalent to $N_{\bullet} \mathcal{C}_{d,n}^k$, the delooping of τ is achieved by extending its definition to a multi-simplicial map

$$\tau^k : \sin_{\bullet} N_{\bullet}^{\delta} \mathcal{C}_{d,n}^k \rightarrow S_{\bullet}^{(k)} W(G(d, n), \Delta^{\bullet}) \quad (23)$$

for $k \leq d+n$.

The construction of τ^k is completely similar to the case of $k=1$; we give the details for $k=2$. Let $a_0 < a_1 < \dots < a_p$ and $b_0 < b_1 < \dots < b_q$ be two sequences of real numbers. Write

$$\begin{aligned} J_{i,j} &= [a_0, a_i] \times [b_0, b_j] \subset \mathbb{R}^2 \\ \partial_0 J_{i,j} &= \{a_0\} \times [b_0, b_j] \cup [a_0, a_i] \times \{b_0\}. \end{aligned}$$

Given a smooth singular simplex $\sigma : \Delta^s \rightarrow N_{p,q}^{\delta} \mathcal{C}_{d,n}^2$ we get sequences $\underline{a} = (a_0 < a_1 < \dots < a_p)$, $\underline{b} = (b_0 < b_1 < \dots < b_q)$ that do not vary with $z \in \Delta^s$ and

$$E^{p+q} \subset \Delta^s \times J_{p,q} \times \text{int}(I^{d+n-2}).$$

Let $E_{i,j} = E \cap \Delta^s \times J_{i,j} \times \text{int}(I^{d+n-2})$. It is a compact manifold with corners and the projection $E_{i,j} \rightarrow \Delta^s$ is a smooth fiber bundle where tangents along the fibers give compatible maps from $E_{i,j}$ to $G(d, n)$. Form

$$Y_{i,j} = E_{i,j} \cup_{\partial_0 E_{i,j}} G(d, n) \times \Delta^s \in W(G(d, n), \Delta^s)$$

¹A weak map from X to Y is a composite of the form $X \leftarrow X' \rightarrow Y$ with $X' \rightarrow X$ a weak homotopy equivalence, for instance an invertible map in the homotopy category associated to a model category defining the weak homotopy equivalences.

where $\partial_0 E_{i,j} = E_{i,j} \cap \partial_0 J_{i,j}$. The diagram

$$\begin{array}{ccccccc}
Y_{1,1} & \longrightarrow & Y_{2,1} & \longrightarrow & \cdots & \longrightarrow & Y_{p,1} \\
\downarrow & & \downarrow & & & & \downarrow \\
Y_{1,1} & \longrightarrow & Y_{2,1} & \longrightarrow & \cdots & \longrightarrow & Y_{p,1} \\
\downarrow & & \downarrow & & & & \downarrow \\
\vdots & & \vdots & & & & \vdots \\
\downarrow & & \downarrow & & & & \downarrow \\
Y_{1,1} & \longrightarrow & Y_{2,1} & \longrightarrow & \cdots & \longrightarrow & Y_{p,1}
\end{array}$$

represents an element of $S_\bullet S_\bullet W(G(n, d), \Delta^s)$, and the resulting map

$$\sin_s N_{p,q}^\delta \mathcal{C}_{d,n}^2 \rightarrow S_\bullet S_\bullet W(G(n, d), \Delta^s)$$

commutes with the simplicial structure maps \cdot . This defines the map (23) for $k = 2$. The general case $k < 2$ is entirely similar.

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